Lecture 16: Bezout's Theorem

Definition 1. *Two (Cartier) divisors are linearly equivalent if D_1 - D_2 are principal.*

Given an effective divisor D, we have an associated line bundle $\mathcal{L} = \mathcal{O}(D)$ given (on each open set U) by the sections of $\mathcal{K}$ whose locus of poles (i.e. locus of zeroes in the dual sheaf) is contained in D. Now suppose X is complete, then given an invertible sheaf $\mathcal{L}$ on X, a section σ is uniquely (up to multiplication by a constant) determined by its corresponding divisor $Z(\sigma)$, so we have a correspondence $D \underset{Z(\sigma)}{\overset{(\mathcal{O}(D),1)}{\longleftrightarrow}} (L, \sigma)$. Now if σ_1, σ_2 are nonzero sections, then $f = \sigma_1/\sigma_2$ is an rational function on X, and if $Z(\sigma_1)$ and $Z(\sigma_2)$ are linearly equivalent, then f has no pole and no zero; in other words, linearly equivalent divisors correspond to isomorphic line bundles. So the set of all effective divisors linearly equivalent to a fixed effective divisor D form a projective space $\mathbb{P}\Gamma(\mathcal{O}(D))$, and is called a *complete linear system of divisors.*

Proposition 1. *X irreducible curve,* $\deg(D) = 0$ *if D is a principal divisor.*

Proof. D is principal, so let $D = (f) = D_0 - D_\infty$ where $f : X \to \mathbb{P}^1$, $X = U_1 \cup U_2$, $f \in k[U_1], 1/f \in k[U_2]$, (This is clear for X normal: all local rings are DVR, so either f or $1/f$ is in $\mathcal{O}_{X,x}$.) where $D_0 \subseteq f(\mathbb{P}^1 - \{\infty\})$ is the divisor of zeroes of f, and similarly $D_\infty \subseteq 1/f(\mathbb{P}^1 - \{0\})$ is the divisor of zeroes of $1/f$. We need to check that degree of D_0 is the same as that of D_∞, and that the degree of both slices are that of $\deg(f)$.

Recall that $D_0 = \sum_{x \in f^{-1}(\mathbb{P}^1 - \{\infty\}), f(x)=0} m_x x$, where $m_x = \text{length}(\mathcal{O}/f\mathcal{O})_x = \dim(\Gamma((\mathcal{O}/f\mathcal{O})_x))$.[1] Clearly $f : U = f^{-1}(\mathbb{A}^1) \to \mathbb{A}^1$ is finite, and that $f_*(\mathcal{O}_X|_U)$ is a locally free sheaf of rank equal to the degree of f. From classification of finitely generated modules over $k[t]$, we know that every module is the sum of its torsion and a free module; but this one cannot have torsion because there can be no function of X that vanishes away from finitely many points, so it's free.

$f_*\mathcal{O}$ is coherent follows from f being finite, which follows from that f is complete and has finite fibers. Now suppose $k[f^{-1}(\mathbb{A}^1)]$ is a free module of rank d over $k[t] = k[\mathbb{A}^1]$. Then $[K(X) : K(\mathbb{A}^1)] = d$, which is the degree of the map. Thus $d = \dim(k[f^{-1}(\mathbb{A}^1)]/t)$ (dimension of fiber of $f_*\mathcal{O}$ at 0) $= \dim(\Gamma(\mathcal{O}_{U_1}/f\mathcal{O}_{U_1})) = \sum \dim(\Gamma((\mathcal{O}_{U_1}/f\mathcal{O}_{U_1})_x)) = \deg(D_0)$, where $U_1 = f^{-1}(\mathbb{A}^1)$. The other half is dealt with similarly. ☐

Remark 1. $k = \mathbb{C}$*, X normal, $X(\mathbb{C})$ (the set X equipped with the complex topology) is a smooth compact Riemann surface (1-dimensional $\mathbb{C}$-manifold). $f \in K(X)$ defines a meromorphic function on $X(\mathbb{C})$, $(f) = \sum n_x x$, n being the order of zero/pole, or just* $\text{Res}_x \frac{df}{f}$*, which tells us that* $\sum_{x \in X(\mathbb{C})} \text{Res}_x \frac{df}{f} = 0$.

Proof of Bezout's Theorem The multiplicity of intersection of two curves X, Y in $\mathbb{P}^2$ at x (X, Y have no common components) is defined as $\text{mult}_x(X, Y) = \text{length}(i_*\mathcal{O}_X \otimes_{\mathcal{O}(\mathbb{P}^2)} j_*\mathcal{O}_Y)_x = \dim\ \Gamma((i_*\mathcal{O}_X \otimes_{\mathcal{O}(\mathbb{P}^2)} j_*\mathcal{O}_Y)_x)$. Note that $(i_*\mathcal{O}_X \otimes_{\mathcal{O}(\mathbb{P}^2)} j_*\mathcal{O}_Y) = \bigoplus_{x \in X \cap Y} (i_*\mathcal{O}_X \otimes_{\mathcal{O}(\mathbb{P}^2)} j_*\mathcal{O}_Y)_x$. This agrees with earlier definition.

Theorem 1.1 (Bezout's Theorem)**.** $\sum_{x \in X \cap Y} \text{mult}_x(X, Y) = \deg(X)\deg(Y)$.

Proof. Both sides are additive under $X = X_1 \cup X_2$ where the two curves have no common components. (Clear for RHS, LHS as exercise.) Now we can assume X is irreducible, and we'll show LHS $= \deg(\mathcal{O}(Y)|_X)$.

$\mathcal{O}(Y)$ is a line bundle with a section σ such that $(\sigma) = Y$. We know that $\mathcal{O}_Y = \mathcal{O}/\mathcal{O}(-Y)$ from which it follows that $\mathcal{O}_X \otimes \mathcal{O}_Y = \mathcal{O}_X/\text{im } \sigma|_X$ (where σ denotes $\mathcal{O}(-Y) \xrightarrow{\sigma} \mathcal{O}$). Compare with the definition of multiplicity above, it follows that the divisor of zeroes of $\sigma|_X$, i.e. the pullback of σ, is $\sum \text{mult}_x(X, Y)x$.

Now we know that $\mathcal{O}(Y) \cong \mathcal{O}(d)$ where $d = \deg(Y)$, so the isomorphism class and hence the degree of $\mathcal{O}(Y)|_X$ depends only on the degree of Y. Now we can take Y to be the union of d lines; by additivity, we reduce to the case where Y is a line. Since Y and X are symmetric, also reduce to X is a line, from which the result follows. ☐

[1] The subscript here refers to the canonical split of sheaves supported at finitely many points, NOT stalks; the same for below.

The analytic story Let X be an irreducible normal curve over $\mathbb{C}$, then $X(\mathbb{C})$ is a compact 1-dimensional $\mathbb{C}$-manifold homeomorphic to a sphere with g handles, g being the genus of the curve. One can look at the topological homology $H^1(X,\mathbb{Z}) = \mathbb{Z}^{2g}$. The important variant here is the space of differential forms. Define Ω^1 to be the sheaf of holomorphic 1-forms, e.g. $f(z)dz$. The global section $\Gamma(\Omega^1) \cong \mathbb{C}^g$. Now, since we have Poincare duality, we can define a map from de Rham classes to singular cohomology as follows: given an 1-form ω, we map it to $\mathrm{Hom}(H_1(X,\mathbb{C}),\mathbb{C}) = H^1(X,\mathbb{C}) = \mathbb{C}^{2g}$ as $[c] \mapsto \int_c \omega$. Thus we have $H^1(X,\mathbb{C}) = \mathrm{Im}(\Gamma(\Omega^1)) \oplus \overline{\mathrm{Im}(\Gamma(\Omega^1))} = H^{1,0} \oplus H^{0,1}$, usually called the *Hodge decomposition.*

Recall the GAGA theorem, which states that holomorphic line bundles are the same as algebraic line bundles, which are parametrized by the Picard group. Now Picard group is (Divisors) / (Principle Divisors), and there is a degree homomorphism $\mathrm{Pic} \to \mathbb{Z}$, with the kernel denoted Pic°. It turns out that $\mathrm{Pic}^\circ \cong \Gamma(\Omega^1)^*/H_1(X,\mathbb{Z})$ (image of $H_1(X,\mathbb{Z}) \subseteq H_1(X,\mathbb{C})$ under the integral map) $\cong \mathbb{C}^g/\mathbb{Z}^{2g}$. The structure $\Gamma(\Omega^1)^*/H_1(X,\mathbb{Z})$ is usually called the *Jacobian* of the curve, and the isomorphism the *Abel-Jacobi map.*

If $D = (f)$ is a principal divisor, D gets mapped into 0 by the Abel-Jacobi map above. Sketch of proof: given f from $X \to \mathbb{P}^1$, consider a family of divisors $D_0 - D_z$, $z \in \mathbb{P}^1$. If $z = 0$, then this is the 0 divisor; when $z = \infty$, we get our divisor $D = (f)$. Easy to see that $z \mapsto AJ(D_0 - D_z)$ is a holomorphic function $\mathbb{CP}^1 \to \mathbb{C}^g/\mathbb{Z}^{2g}$. Since $\mathbb{CP}^1$ is simply connected, it lifts to $\mathbb{CP}^1 \to \mathbb{C}^g$, which is constant by maximal principle.

Our next topic is smoothness, which is a local property. Let X be an algebraic variety, and x be a point. Define $\dim_x(X)$ to be the maximum of dimensions of components passing through x.

Definition 2. *x is a smooth point on X if $\dim_x(X) = \dim(\mathfrak{m}_x/\mathfrak{m}_x^2)$, where $\mathfrak{m}_x$ is the maximal ideal in $\mathcal{O}_{X,x}$.*

Example 1. *Suppose X in $\mathbb{A}^n$ is a hypersurface (so codimension 1), $I_X = (f)$. Then x is a smooth point iff $\partial f/\partial z_i \neq 0$ at x for some i.*

Corollary 1. *For X, Y curves in $\mathbb{P}^2$, the intersection multiplicity is greater than 1 if either X or Y is not smooth at x.*

To see this, suppose $x = (0,0) \in \mathbb{A}^2$, then $\mathcal{O}_X \twoheadrightarrow k[x,y]/(x,y)^2$, then $\mathcal{O}_X \otimes \mathcal{O}_Y \twoheadrightarrow \mathcal{O}_Y/\mathfrak{m}^2_{\mathcal{O}_Y}$.

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