Lecture 18: Kähler Differentials

Last time we proved that principal divisors on a complete normal curve has degree zero. This actually remains true for Cartier divisors on irreducible non-normal curves. To prove this, we show that the degree of a divisor is preserved under pull-back to normalization. Let D be a principal divisor on a non-normal irreducible curve X. We may assume that $D = (f)$ is supported at a point x, the curve is complete and normal away from x, so that f defines a map $X \to \mathbb{P}^1$. The total degree of the divisor of zeroes of f is the same on X and on the normalization $Nm(X)$, both are equal to the degree $deg(\tilde{f})$, where $\tilde{f}$ is the composition $Nm(X) \to X \to \mathbb{P}^1$.

Today we begin the discussion of tangent and cotangent spaces and smoothness. The first step is to define (Kähler) differentials.

Definition 1. *Let A be a commutative k-algebra. Ω_A is defined to be the A-module generated by expressions $da, a \in A$, modulo the following equations:*

- $d(a+b) = da + db$;
- $d(\lambda a) = \lambda da$;
- $d(ab) = (da)b + a(db)$,

where $a, b \in A$, $\lambda \in k$. Then Ω_A is characterized by a universal property: $Hom(\Omega_A, M) = Der(A, M)$ for any A-module M, where $Der(A, M)$ is the k-module of k-linear derivations from A to M.

As an alternative way to define Ω_A, suppose that A is generated by $a_1, \ldots, a_n$. Let $X = \text{Spec } A$ and I_m be the ideal of X in the diagonal $X \subset X \times X$. Then $(a_i \otimes 1 - 1 \otimes a_i)$ generate $I_m \subset A \otimes A$. Therefore Ω_A is finitely generated. This approach also allows us to define a coherent sheaf Ω_X on X, called the sheaf of differentials on X.

Let $f : A \to B$ be a morphism of rings. Then there is a canonical morphism $B \otimes_A \Omega_A \to \Omega_B$ given by $da \mapsto d(fa)$. Let $Y = \text{Spec } B$. Then this morphism of rings gives rise to the morphism of varieties $Y \to X$, $df : f^*\Omega_X \to \Omega_Y$.

Now for an arbitrary variety X over k, we may define the sheaf Ω_X by gluing the above constructions on affine charts. Then it is straightforward to check that $Hom(\Omega_X, \mathcal{F}) = Der(\mathcal{O}_X, \mathcal{F})$, where $\mathcal{F}$ is a coherent sheaf on X, and $Der(\mathcal{O}_X, \mathcal{F})$ is the set of k-linear derivations $\mathcal{O}_X \to \mathcal{F}$, i.e. sheaf morphisms satisfying Leibniz rule on each chart.

Definition 2. *Let X be a variety. The Zariski cotangent space of X at $x \in X$ is defined to be the vector space $\{\xi : \mathcal{O}_{X,x} \to k \mid \xi \text{ is linear and } \xi(fg) = f(x)\xi(g) + g(x)\xi(f)\}$, i.e. it is the set of derivations at x, and it is denoted as T_x^*X.*

One can check that $(\Omega_X)_x = T_x^*X$.

Now we define the tangent sheaf $\mathcal{T}_X$ on X as $\mathcal{T}_X = Hom(\Omega_X, \mathcal{O}_X)$. Note however that even though there is always a map $\Omega_X \to Hom(\mathcal{T}_X, \mathcal{O}_X)$, it is not neccessarily an isomorphism.

Lemma 1. $\dim(T_x^*X) \geq \dim_x(X)$.

Proof. We may assume $X = \text{Spec } A$ and $\mathfrak{m}$ the maximal ideal corresponding to x. Let $df_1, \cdots, df_n$ be the generators of $\mathfrak{m}/\mathfrak{m}^2$, where each df_i is lifted to $f_i \in \mathfrak{m}$. By Nakayama lemma, f_i generate $\mathfrak{m}$. Now, as a consequence of the hypersurface theorem, $\dim_x X \leq n$. □

Definition 3. *Let $x \in X$. X is said to be smooth at x if* $\dim(T_x^*X) = \dim_x(X)$.

Proposition 1. *X is smooth at $x \in X$ if and only if Ω_X is locally free on a neighborhood of x.*

Proof. One direction (from right to left) will follow from the next proposition. For the other direction (from left to right), recall the lemma stated during the lecture on October 22th, asserting that if all fibers of a coherent sheaf have the same dimension, then the sheaf is locally free, combined with the fact (that we will prove next time) that smooth varieties are locally irreducible. □

Proposition 2. *For a variety X, the set of smooth points in X is open and dense in X.*

Proof. It follows from the previous proposition(left to right) that the set of smooth points in X, which we denote by X_{sm}, is open in X. Now, to prove that X_{sm} is dense in X, we may assume X is affine and irreducible, and is actually embedded as a closed subset $X \subset \mathbb{A}^n$. Let $d = n - \dim X$. We proceed by induction on d. If $d = 0$ then $X = \mathbb{A}^n$, which is smooth everywhere, and there is nothing to prove. Now for $d > 0$, we may find $g \in k[\mathbb{A}^n]$ vanishing on X. choose g to have minimal degree among such functions. We claim that $\frac{\partial g}{\partial x_i}$ is not identically zero on X for at least one x_i. To see this, suppose to the contrary that $\frac{\partial g}{\partial x_i}$ is identically vanishing on X. If $char k = 0$, by the minimality of degree of g, this means g is a constant function which is not zero. Then g cannot vanish on X, a contradiction. if $char k = p$, then replacing g with $g^{1/p}$ gives a function identically vanishing on X with a smaller degree than g, a contradiction. Hence the claim holds. After a change of coordinate, we may assume that g is monic in x_n and $\frac{\partial g}{\partial x_n}$ is not identically zero on X. now, consider the projection $\pi : \mathbb{A}^n = \text{Spec } k[x_1, \cdots, x_n] \to \mathbb{A}^{n-1} = \text{Spec } k[x_1, \cdots, x_{n-1}]$. Let Y be the image of X under this projection. Then since π is finite, $\dim Y = \dim X$. Since Y is a closed subset of $\mathbb{A}^{n-1}$ we may apply the induction hypothesis on Y, so that the smooth points of Y consist an open and dense subset of Y. Now we claim that if $x \in X$ is such that $\frac{\partial g}{\partial x_n} \neq 0$ at x and $\pi(x)$ is a smooth point of Y, then X is smooth at x. Indeed, for such x, $\pi : X \to Y$ induces a surjection $T^*_{\pi(x)}Y \oplus (gdx_n|_x)/dg|_x \to T^*_x X$. Therefore, $\dim T^*_x X \leq \dim T^*_y Y = \dim Y = \dim X$. By a previous lemma, $\dim T^*_x X = \dim X$. Hence x is a smooth point of X. The set of all such x is dense in X, hence X_{sm} is dense in X. □

Remark 1. *A curve is defined to be a variety of dimension one. For a curve X, the following are equivalent:*

- *X is smooth.*
- *All the local rings of X are DVR(=discrete valuation ring)s.*
- *X is normal.*

Remark 2. *As a final remark, let X be a hypersurface in $\mathbb{A}^n$ with $I_X = (f)$. Let $x \in X$. Then X is smooth at x if and only if I_X is locally generated by some $f_1, \cdots, f_m$ such that rank$(\frac{\partial f_i}{\partial x_j}) = m$. This is also equivalent to saying that $\widehat{\mathcal{O}_{X,x}} := \varprojlim_n \mathcal{O}_{X,x}/\mathfrak{m}_x^n \cong k[[x_1, \cdots x_m]]$.*

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