Lecture 20: (Co)tangent Bundles of Grassmannians

Last time we proved that $X \subseteq \mathbb{A}^n$ is smooth at x if and only if locally given by equations $f_1, \ldots, f_m$ such that $df_i|_x$ are linearly independent. We say that $\mathscr{I}_X$ is locally generated by $f_1, \ldots, f_m$. In fact, any $f_1, \ldots, f_m$ such that $df_i|_x$ is a basis for $\ker(T_x^*\mathbb{A}^n \to T_x^*X)$ would work. Take Z generated by the equations $f_1, \ldots, f_m$. We checked that $\dim_x(Z) = \dim_x(X)$.

Proposition 1. *The following hold:*

1. *If $Z \subseteq X$ is a closed subvariety, then we have $\mathscr{I}_Z/\mathscr{I}_Z^2 \to \Omega_X|_Z \to \Omega_Z \to 0$.*
2. *If $\mathscr{I}_Z$ is locally generated by functions with linearly independent differential (that is, for all x in Z, there exists $U \ni x, f_1, \ldots, f_m$ on U such that $\mathscr{I}_{Z\cap U} = (f_1, \ldots, f_m), df_i|_y$ is linearly independent for any $y \in U$), then the sequence is exact at left.*
3. *If X is smooth, the last condition can be checked at x. (Ω_X is locally linearly independent of $df_i|_x$ is an open condition.)*

Proof. 1. $\Omega_X|_Z$ surjects to Ω_Z by sending fdg to $f|_Z dg|_Z$, and we claim that the kernel is generated by $fg, g \in \mathscr{I}_Z$. This would follow from $\mathrm{Der}(\mathcal{O}_Z, M) = \{\delta \in \mathrm{Der}(\mathcal{O}_X, M) \mid \delta(\mathscr{I}_Z) = 0\}$, so it remains to see that $f \mapsto df|_Z$ is a well-defined map of $\mathcal{O}_Z$ mod $\mathscr{I}_Z/\mathscr{I}_Z^2 \to \mathcal{O}_X|_Z$. Observe that $f, g \in \mathscr{I}_Z \implies d(fg)|_Z = 0$.

2. If $\mathscr{I}_Z = (f_1, \ldots, f_m)$, we have the following diagram:

$$\begin{array}{ccc} \mathscr{I}_Z/\mathscr{I}_Z^2 & \longrightarrow & \Omega_X|_Z \\ \uparrow\!\!\uparrow & \nearrow & \\ \mathcal{O}_Z^{\oplus n} & & \end{array}$$

where the diagonal map is guaranteed to be injective on every fiber by condition b), so is injective.

3. We always have it for affine space $\mathbb{A}^n$. General case is proved similarly.

□

Corollary 1. *X smooth, $Z \subseteq X$ closed, then Z is smooth if and only if locally Z is given by equation with linearly independent differentials.*

Proof. Use proposition 3) above. Locally we assume $X \subseteq \mathbb{A}^n$, and then X is cut out by some $g_1, \ldots, g_p$ with linearly independent differentials, so $(g_1, \ldots, g_p, \tilde{f}_1, \ldots, \tilde{f}_n)$ are equations for Z with linearly independent differentials, so Z is smooth. □

Last time we defined ω, the canonical bundle. Let K be the corresponding canonical divisor class.

Corollary 2. *If X, Z smooth, Z closed in X, then we get a s.e.s. of locally free sheaves $0 \to \mathscr{I}_Z/\mathscr{I}_Z^2 = T_Z^*X \to \Omega_X|_Z \to \Omega_Z \to 0$, and thus $K|_Z = K_Z\omega(\mathscr{I}_Z/\mathscr{I}_Z^2)$. If Z is a divisor, then $\omega(\mathscr{I}_Z/\mathscr{I}_Z^2) = \mathscr{I}_Z/\mathscr{I}_Z^2 = \mathcal{O}(-D)|_Z$, thus $K_X(D)|_D = K_D$, which is the adjunction formula.*

Remark 1. *Sections of $K_X(D)$ are top degree forms on X with poles of order ≤ 1 on D. The map $K_X(D)|_D \to K_D$ sends ω to its residue.*

Proposition 2. *We have a s.e.s. $0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}(-1)^{\oplus(n+1)} = \mathcal{O}(-1) \otimes V^* \to \mathcal{O} \to 0$ where $\mathbb{P}V = \mathbb{P}^n$. As a corollary, $K_{\mathbb{P}^n} = \mathcal{O}(-(n+1))$.*

More generally, consider the Grassmannian $\mathrm{Gr}(k, n)$, consisting of all k-dimensional linear subspaces V of an n-dimensional space W. Then $\mathcal{O}_{\mathrm{Gr}(k,n)}^{\oplus n}$ has a locally free *tautological subsheaf* $\mathcal{V}$ of rank k (that is locally a direct summand) such that a section s of $\mathcal{O} \otimes W$, i.e. a map $s : Gr(k, n) \to W$, belongs to $\mathcal{V}$ if for all x, $s(x) \subseteq \mathcal{V}_x$.

Proposition 3. *$T_{Gr(k,n)} = \mathrm{Hom}(\mathcal{V}, W \otimes \mathcal{O}/\mathcal{V})$ and $\Omega_{Gr(k,n)} = \mathrm{Hom}(W \otimes \mathcal{O}/\mathcal{V}, \mathcal{V})$.*

Let's see how this implies the last proposition: let $k = 1, \mathcal{V} = \mathcal{O}(-1)$. Then $\mathrm{Hom}\left(\mathcal{O}(-1), \frac{\mathcal{O}^{\oplus(n+1)}}{\mathcal{O}(-1)}\right) = \frac{\mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}^{\oplus(n+1)})}{\mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}(-1))} = \frac{\mathcal{O}(1)^{\oplus(n+1)}}{\mathcal{O}}$ and $\Omega = \ker(\mathcal{O}(-1)^{n+1}, \mathcal{O})$.

Proof of the Second Proposition. For any point V on $\mathrm{Gr}(k,n)$, we have an isomorphism $T_V\mathrm{Gr}(k,n) \cong \mathrm{Hom}(V, W/V)$ by identifying a neighborhood of V with $\mathrm{Hom}(V, V')$. Check this is independent of the choice of V', so let $V' = W/V$, and glue together these open charts. □

Second Proof of the First Proposition. It suffices to construct an s.e.s. of sheaves on $\mathbb{A}^{n+1} - \{0\}$ that is compatible with the G_m action. Let $\pi : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$, and consider the s.e.s. $0 \to \pi^*\Omega_{\mathbb{P}^n} \to \Omega_{\mathbb{A}^{n+1}-\{0\}} \to \mathcal{O} \to 0$. See [Kem93] for more details. □

Application Let $X \subseteq \mathbb{P}^n$ be a smooth hypersutrface of degree $d = n+1$, then $K_X \cong \mathcal{O}_X$ is trivial. (Proof: $K_X = K_{\mathbb{P}^m}(X)|_X = \mathcal{O}(-(n+1)+d)|_X$.)

Here are some examples of X:

1. $n = 2, d = 3$. This gives us the elliptic curves.
2. $n = 3, d = 4$. These are the K3 surfaces.
3. $n = 2$, $d =$ any. We see that the degree of the canonical class is $\deg(K_X) = \deg(\mathcal{O}(-3+d)|_X) = d(d-3)$. Recall that complete smooth curves have genus as an invariant, such that $\deg(K_X) = 2g - 2$, so we have $g = d(d-3)/2 + 1$.

Now let X be an affine variety, $X = \mathrm{Hom}(k[X], k)$. We can write the tangent bundle as $TX = \coprod_{x \in X} T_xX = \mathrm{Hom}(k[X], k[\varepsilon]/\varepsilon^2) = \mathrm{Hom}(\mathrm{Spec}(k[\varepsilon]/\varepsilon^2), X)$ where the first object, $\mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$, is a scheme rather than a variety. [1] Each such homomorphism $h : k[X] \to k[\varepsilon]/\varepsilon^2$ is given by $f \mapsto h_0(f) + \varepsilon h_1(f)$, where $h_0 : k[X] \to k$ is given by $h_0(f) = f(x)$ for some x, and $h_1 : f \to k$ is a derivation where the target k is made a $k[X]$-module by evaluation at x, i.e. if $h_0(f) = f(x)$ then $h_1(fg) = f(x)h_1(g) + g(x)h_1(x)$.

Proposition 4. *Let E be the exceptional locus over x when blowing up $X \ni x$. Then the cone of E is the same as $\mathrm{Spec}(\bigoplus_{n \geq 0} \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1})_{\mathrm{red}}$, which we call the tangent cone. If we know that x is a smooth point, then $\bigoplus_{n \geq 0} \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$ is given by $\mathrm{Sym}(T_x^*X)$.*

Proof. Let $A = k[x_1, \ldots, x_n]$, then it surjects to $\bigoplus_{n \geq 0} \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1} = \mathrm{gr}_x(A)$ (the *associated graded ring*). So $\mathrm{Cone}(E)$ and $\mathrm{Spec}(\mathrm{gr}_x(A))$ both sit above $\mathbb{A}^n$, so let's compare their associated ideals. We can do it on each of the affine coverings for $E \subset \mathbb{P}^{n-1}$, which has coordinates, say, $(\lambda, t_1, \ldots, t_n)$ (this is for $\mathbb{A}^n_0$) such that the map to $\mathbb{A}^n$ is generated by $(\lambda, t_1, \ldots, t_n) \mapsto (\lambda, \lambda t_1, \ldots, \lambda t_n)$. The ideal of $E \cap \mathbb{A}^n_0$ is generated by polynomials $P(\lambda, \lambda t_1, \ldots, \lambda t_n)/\lambda^d$ evaluated at $\lambda = 0$ (where d is the highest degree of λ divisible by $P(\lambda, \lambda t_1, \ldots, \lambda t_n)$), where $P \in \mathscr{I}_X$. We need to compare those with $\ker(A \to \mathrm{gr}_x(A))$: invert x_1 and take the degree 0 part, we see the latter is generated by $\{P_d \mid P = P_d + P_{d+1} + \ldots \in \mathscr{I}_X\}$. □

References

[Kem93] George Kempf. *Algebraic varieties.* Vol. 172. Cambridge University Press, 1993.

[1] There was a question why $k[\varepsilon]/\varepsilon^2$ was called the *dual* number; answer: dual refers to the fact that there are *two parts* of each element.

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