Lecture 21: Riemann-Hurwitz Formula, Chevalley's Theorem

We begin with a remark on the tangent cone. Let X be a variety and $x \in X$.

- We checked that $Spec((\oplus \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1})_{red})$ is the tangent cone over $\pi^{-1}(x) \subset \mathbb{P}^n$, where $\pi : \hat{X} \to X$ is the blow-up of X at x. If $X = \text{Spec } A$ we can do this for any ideal in A; indeed, applying it to $\mathscr{I}_Z$, where $Z \subset X$ is a closed subvariety, we get that the "normal cone" to Z is Spec $((\oplus \mathscr{I}_Z^n/\mathscr{I}_Z^{n+1})_{red})$. Using the relative Spec, we can generalize this to non-affine case. If X and Z are smooth then we get the total space of the normal bundle.

- X can be degenerated into the normal cone, i.e. there is a morphism of varieties $\tilde{X} \to \mathbb{A}^1$ which satisfies the following situation:

$$\begin{array}{ccccc} N_X(Z) & \longrightarrow & \tilde{X} & \longleftarrow & X \times (\mathbb{A}^1 \setminus \{0\}) \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathbb{A}^1 & \longleftarrow & \mathbb{A}^1 \setminus \{0\} \end{array}$$

  Compare this with the fact that a filtered space can be degenerated into its associated graded ring:

$$\{\text{A locally free coherent sheaf on } \mathbb{A}^1 \text{equivariant with respect to } \mathbb{G}_m\} \leftrightarrow \{\text{filtered vector spaces}\}.$$

  To describe the equivalence, let $\mathcal{E}$ be a locally free coherent sheaf on $\mathbb{A}^1$ corresponding to a module M over $k[t]$, and V be a filtered vector space. Then the equivalence is given by $\mathcal{E} \mapsto \mathcal{E}_1 = M/(t-1)M$ with the filteration $(\mathcal{E}_1)_i = \text{im}(M_i \to M/(t-1)M)$ and $V = V_j \supset \cdots \supset V_{i+1} \supset V_i = 0$ $(i \ll 0, j \gg 0) \mapsto M$, $M_i = V_{\leq i}$.

Theorem 1.1 (Riemann-Hurwitz formula)**.** *Let $f : X \to Y$ be a morphism of smooth irreducible curves. Then $k(X)/k(Y)$ is a separable extension.*

Recall from the lecture on September 22th, that for $x \in X$ we have the ramification index d at x if the divisor $f^{-1}(f(x))$ has coefficient of the irreducible divisor x equal to d. This is equivalent to saying that in the extension of DVRs $\mathcal{O}_{y,Y} \subset \mathcal{O}_{x,X}$, $(\text{val}_{\mathcal{O}_{x,X}})|_{\mathcal{O}_{y,Y}} = d \cdot \text{val}_{\mathcal{O}_{y,Y}}$.

Let d_x be the ramification index at x. Assume that d_x is prime to char(k). Then $f^*K_Y \to K_X$ extends to an isomorphism $f^*K_Y(R) \simeq K_X$ where $R = \sum_{x \in X}(d_x - 1)x$.

Corollary 1. *If X, Y are complete then* $\deg K_X = n \cdot \deg K_Y + \sum_{x \in X}(d_x - 1)$.

Let's consider the example of elliptic curves. Let X be the projective plane curve defined by the equation $y^2 = x^3 + ax + b$. Then the projection $(x : y) \mapsto x$ extends to a map $X \to \mathbb{P}^1$, which is ramified at the roots of the polynomial $P(x) = x^3 + ax + b$ and the point at infinity ∞, with a unique point over each ramification point. Moreover, from the adjunction formula, $\deg K_{\mathbb{P}^1} = -2$. Therefore, $\deg K_X = 2(-2) + 4 = 0$. Observe that if $x \in X$ is a smooth point on a curve and f is a function on X not equal to 0 with $f(x) = 0$, then $\frac{df}{f}$ has a pole of order exactly 1 at x, i.e. it is a local generator of $K_X(x)$ (an exception is when char $k = p$, $(f) = n_x(x) + (\text{other points})$, $p|n_x$). If $f \in \mathfrak{m}_x/\mathfrak{m}_x^2$ then df is a local generator for $K_X \simeq \Omega_X$. In general, if $f = \varphi g^n$ where $\varphi(x) \neq 0$ and $g \in \mathfrak{m}_x/\mathfrak{m}_x^2$, then $d \cdot \deg f = d \cdot \deg \varphi + dn_x \cdot \deg g$. Now, take $f \in \mathfrak{m}_x \subset \mathcal{O}_{x,X}$. Then $f^*K_Y \to K_X$ extends to a local isomorphism $f^*K_Y(y) \simeq K_X(x)$, where $f^*K_Y(y) = f^*K_Y \otimes f^*y$ and similarly for $K_X(x)$. Therefore, $f^*K_Y(R) \simeq K_X(\sum_{d_x > 1} x)$.

Recall that a smooth irreducible variety is normal, but the converse is true only in dimension 1.

Proposition 1. *Let X be a normal irreducible affine variety and $X \subset X$ be a closed subvariety. If* $\dim Z \leq \dim X - 2$ *then $k[X] = k[X \setminus Z]$. Therefore, for normal varieties, the regular functions extend from the complement of a codimension ≥ 2 closed subvariety to the whole space.*

Proof. We may assume that X is irreducible. Using induction on $\dim Z$ we can reduce to showing that any $f \in k[X \setminus Z]$ is regular generically on Z, i.e. there exists an open subset $U \supset X \setminus Z$ such that f is regular on U. Suppose that this is not true for some $f \in k[X \setminus Z]$. Then f generates a coherent sheaf $\mathscr{F} \subset \mathrm{Rat}(X)$ where $\mathrm{Rat}(X)$ is the sheaf of rational function on X, such that $\mathscr{F}|_{X \setminus Z} \subset \mathcal{O}$. Thus $\mathscr{F}/\mathscr{F} \cap \mathcal{O}$ is coherent, supported on Z, and killed by $\mathscr{I}_Z^m$. After modifying the choice of f we can assume that $m = 1$, i.e. $\mathscr{I}_Z(\mathscr{F}/\mathscr{F} \cap \mathcal{O}) = 0$. Thus, for any $\varphi \in \mathscr{I}_Z$, $\varphi f \in k[X]$, but for any open subset $U \supset X \setminus Z$, $f \notin k[U]$. Now we claim that for any $\varphi \in \mathscr{I}_Z$, $\varphi f \in \mathscr{I}_Z \subset k[X]$. Indeed, by the hypersurface theorem, $\varphi|_D = 0$ for some Weil divisor $D \supset Z$. Suppose that $z \in Z$ and $\varphi f(z) \neq 0$. Then $\varphi f \neq 0$ on some neighborhood U of z and, by assumption on f, f is not regular on $D \cap U$, a contradiction. Hence, $\varphi f \in \mathscr{I}_Z$. By replacing φ with φf, we obtain that $\varphi f^2 \in \mathscr{I}_Z$. Using induction we conclude that $\varphi f^n \in \mathscr{I}_Z$. To get a contradiction it is enough to check that $\{f^n\}$ generates a finite $\mathcal{O}_X$-module. But, by the previous argument, $f^n \in \{\psi | \mathscr{I}_Z \psi \in k[X]\} \subset (\varphi f)^{-1} k[X]$, the last one being a finite $\mathcal{O}_X$-module. Therefore $\{f^n\}$ generates a finite $\mathcal{O}_X$-module, finishing the proof. ☐

Note that the normality assumption in the above proposition is necessary: Let $A = \{a_0 + a_2 P_2 + a_3 P_3 + \cdots\}$, where P_i is a homogeneous polynomial in n indeterminates of degree i. Then $\mathrm{Spec}(A)$ is non-normal with the normalization $\mathbb{A}^n \to X = \mathrm{Spec}(A)$, which is bijective and an isomorphism away from zero. However, $A = k[X] \neq k[X \setminus \{0\}] = k[\mathbb{A}^n \setminus \{0\}]$.

We say a set is *constructible* if it is a finite union of locally closed subvarieties of Y.

Theorem 1.2 (Chevalley's theorem). *Let $f : X \to Y$ be a morphism of varieties. Then:*

- $\mathrm{im}(f)$ *is constructible.*
- *Furthermore, if we assume that X, Y are irreducible and that $\mathrm{im}(f)$ is dense in Y, then the function on $\mathrm{im}(f)$ given by $f(x) \mapsto \dim f^{-1}(f(x))$ (the dimension of the fiber) is upper semi-continuous. In other words, for any d, $\{f(x) | \dim f^{-1}(f(x)) \geq d\}$ is close in $\mathrm{im}(f)$.*
- *Finally, under the previous assumptions, there exist a non-empty open subset U in Y such that $\dim f^{-1}(y) = \dim X - \dim Y$ for all $y \in U$.*

Lemma 1. *Let $f : X \to Y$ be a morphism of irreducible affine varieties with $\mathrm{im}(f)$ dense in Y. Then there is a nonempty open subset $U \in Y$ such that $f^{-1}(U) \to U$ factors as $f : f^{-1}(U) \xrightarrow{\text{finite,onto}} U \times \mathbb{A}^n \xrightarrow{\pi_1} U$.*

Proof. Let $\mathcal{K}$ be the fraction field of $k[Y]$. Consider $k[X] \otimes_{k[Y]} \mathcal{K}$ which is finitely generated over $\mathcal{K}$ and has no nilpotents. We can apply Noether normalization lemma to find $f_1, \cdots, f_n \in k[X] \otimes \mathcal{K} = A$ such that A is finite over $k[f_1, \cdots, f_n]$. Let $\{g_i\}$ be generators of $k[X]$. The $\{g_i\}$ must satisfy monic equations over $k[f_1, \cdots, f_n]$. We can now choose U so that all f_i and the coefficients of the equations are in $k[U]$. ☐

The lemma implies that if f has a dense image, then $\mathrm{im}(f)$ contains a dense affine open subset.

Proof of Chevalley's theorem. The first part of the theorem now follows from the implication of the lemma, and by Noetherian induction. To prove the remaining, we can assume, without loss of generality, that X, Y are both affine. By the lemma, obtain an open subset U in Y such that $\dim f^{-1}(y) = \dim X - \dim Y$, $\forall y \in U$. Use the hypersurface theorem and induction on $\dim Y$ to conclude that the dimension of every nonempty fiber is at least $\dim X - \dim Y$. Now, using Noether normalization for Y, obtain a finite surjective morphism $g : Y \to \mathbb{A}^m$ where $m = \dim Y$. Let $z \in \mathbb{A}^m$ and $y \in \mathrm{im}(f) \cap g^{-1}(z)$. Then the fiber $f^{-1}(y)$ is a union of components of $(gf)^{-1}(z)$. By the hypersurface theorem, every such component has dimension $\geq \dim X - m$. ☐

18.725 Algebraic Geometry
Fall 2015