Lecture 23: Derived Functors, Existence of Sheaf Cohomology

Prelude: the cousin problem How do we integrate a rational function $\frac{P(x)}{Q(x)}$? We decompose it into a sum $\sum \frac{a_i}{(x-b_i)^{d_i}}$ + polynomial. Conversely, given a complete curve X, and a locally free sheaf $\mathcal{E}$, one may want to understand if $\mathcal{E}$ has a section with singularities at some fixed $x_1, \dots, x_n$ with fixed prescribed singular terms of $x_1, \dots, x_n$. To be more specific, $\sigma \in \Gamma(\mathcal{E}|_{X-\{x_1,\dots,x_n\}}) = \Gamma(j_*j^*\mathcal{E})$ where $j : X - \{x_1, \dots, x_n\} \to X$, and by singular term we mean a section of $j_*j^*\mathcal{E}/\mathcal{E}$, which is a quasicoherent sheaf supported at $x_1, \dots, x_n$. Or one can write $\sigma \in \Gamma(\mathcal{E}(D))$ where $D = \sum_i d_i x_i$, and the singular term is given by a section of $\mathcal{E}(D)/\mathcal{E}$.

This problem can be solved using cohomology. For instance, let $\mathcal{E} = K_X$ be the canonical class, X being smooth irreducible. For instance, let $X = \mathbb{P}^1$, and $x_1 = 0, x_2 = \infty$. Consider the form that takes the shape $\frac{dz}{z}$ + (regular at 0), and $2\frac{dt}{t}$ + (regular at ∞). Can such form exist? No. This follows from Stoke's theorem, which basically says $\sum_x \mathrm{res}_x \omega = 0$. However, in fact for $\mathcal{E} = K_X$ this is the only obstruction: this follows from the fact that $H^1(K_X)$ is one-dimensional.

Back to the main topic Last time we talked about universal δ-functors $\mathcal{R}^i\mathcal{F}$ for a given functor between abelian categories.

Proposition 1 (Grothendieck)**.** *A δ-functor $(\mathcal{F}^i)$ for given $\mathcal{F}$ is universal provided that $\mathcal{F}^i$ for $i > 0$ is effaceable: for any $M \in A$ and any $m \in F^iM$, there exists some monomorphism $\varphi : M \to N$, such that $\mathcal{F}^i(\varphi)(m) = 0$.*

In practice, we often check the stronger condition that $\exists \varphi : M \hookrightarrow N$, such that $\mathcal{F}^i(\varphi) = 0$. Or even stronger one: there exists N such that $\mathcal{F}^i(N) = 0$.

Let X be a separated algebraic variety. Fix an affine open cover $X = U_1 \cup \ldots \cup U_n$. Recall that we have $0 \to \Gamma(F) \to \bigoplus_i \Gamma(F|_{U_i}) \to \bigoplus_{i,j} \Gamma(F|_{U_i \cap U_j})$. This can be extended to a Čech complex $\check{C}(F)$ of the covering:

$$0 \to \bigoplus_i \Gamma(F|_{U_i}) \to \ldots \to \bigoplus_{i_1<\ldots<i_k} \Gamma(F|_{U_{i_1}\cap \ldots \cap U_{i_k}}) \to \ldots$$

with the obvious map having the necessary sign change. One can easily check this is a complex and thus defines a functor $\mathbf{QCoh}(X) \to \mathbf{Complexes}$, which is exact by exactness of Γ on $\mathbf{QCoh}(X)$.

Proposition 2 (Snake Lemma)**.** *A short exact sequence of complexes yields a long exact sequence of cohomology (see Wikipedia for the exact statement).*

We also mentioned that $H^0(\check{C}(\mathscr{F})) = \Gamma(X, \mathscr{F})$. Now we claim that $\mathscr{F} \mapsto H^i(\check{C}(\mathscr{F}))$ is an universal δ-functor. Let's show it's effeceable. Let $j_i : U_i \to X$. Consider the embedding $\mathscr{F} \hookrightarrow \bigoplus_i j_i^* j_{i*}\mathscr{F}$, where we denote the latter object by $\mathscr{G}$. Claim: $H^i(\check{C}(\mathscr{G})) = 0$ for $i > 0$ (reads: $\check{C}(\mathscr{G})$ is *acyclic*). Note that $\Gamma_{i_1,\dots,i_k}(\mathscr{G}) \xrightarrow{\sim} \Gamma_{i_1,\dots,i_k,n}(\mathscr{G})$ when $i_k \neq n$. So $\check{C}(\mathscr{F})$ contains a subcomplex $\check{C}' = \bigoplus \Gamma_{i_1,\dots,i_k|i_k=n}$, and we have a quotient complex $\check{C}''$ given by $\bigoplus \Gamma_{i_1,\dots,i_k|i_k<n}$. Then we have a s.e.s. $\check{C}'(\mathscr{G}) \to \check{C}(\mathscr{G}) \to \check{C}''(\mathscr{G})$, to which if you apply Snake lemma, then the connecting homomorphism will be an iso, thus yielding that the central one is acyclic. (This follows from the observation that $\check{C}(\mathscr{G}) = \mathrm{Cone}(\check{C}'' \to \check{C}'[1])$.) Thus $\mathcal{R}^i\Gamma(\mathscr{F}) = \mathrm{H}^i(\check{C}(\mathscr{F}))$ for any quasicoherent sheaf F.

Remark 1. *More generally, we can use a similar construction with the Čech complex that is the direct limit over all coverings. A theorem of Grothendieck's states that if X is paracompact, then this computes the cohomology for any sheaf F.*

Example 1. *Let X be an algebraic variety. Let $\mathscr{F} = \mathcal{O}^*$ be the sheaf of invertible regular functions. Let's consider $\mathrm{H}^1(\mathcal{O}^*)$. First fix an covering $X = \bigcup U_i$. Then consider the set $f_{ij} \in k[U_i \cap U_j]^*$ such that on*

$U_i \cap U_j \cap U_k$, $f_{ij}f_{jk} = f_{ik}$, modulo $f_{ij} = \varphi_i \varphi_j^{-1}, \varphi_i \in k[U_i]^*$. This defines an invertible sheaf on X. Modulo proof, we know that $\mathrm{H}^1(X, \mathcal{O}^*) \cong \mathrm{Pic}(X)$.

Remark 2. *For any $\mathscr{F}$ and any covering U_i, there exists a canonical map $H^i(\check{C}(\mathscr{F})) \to H^i(\mathscr{F})$.*

Remark 3. *We have the following:*

1. *For $\mathscr{F}$ quasicoherent, $\mathcal{R}^i\Gamma_{\boldsymbol{Sh}(X)}(\mathscr{F}) = \mathcal{R}^i\Gamma_{\mathcal{O}-\boldsymbol{Mod}(X)}(\mathscr{F}) = \mathcal{R}^i\Gamma_{\boldsymbol{QCoh}(X)}(\mathscr{F})$.*

2. *Other relevant derived functors: we have a parallel definition for right exact functors, which then yields $\mathcal{L}^{-i}(\mathcal{F}) = \mathcal{L}_i(\mathcal{F})$ (two different notations) that goes as follows:*

$$\ldots \to \mathcal{L}^{-1}(C) \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \to 0$$

   the case relevant to us is tensor product of modules. For commutative ring A, and a fixed module M, let $\mathcal{F}(N) = M \otimes_A N$, then $\mathcal{L}^{-i}\mathcal{F}(N) = \mathrm{Tor}_i^A(M,N)$. Another funcotr: $f : X \to Y$, then $f^* : \boldsymbol{QCoh}(Y) \to \boldsymbol{QCoh}(X)$. The dual example: fix some $M \in A$ (say $A = \boldsymbol{QCoh}(X)$), and let $\mathcal{F}(N) = \mathrm{Hom}(M,N)$, then $\mathcal{R}^i\mathcal{F} = \mathrm{Ext}^i(M,N)$. For instance for $\mathcal{O}$ the structure sheaf, we have $\mathrm{Ext}^i(\mathcal{O}, \mathcal{F}) = \mathrm{H}^i(\mathcal{F})$.

3. *From a homological point of view, all of $\mathcal{R}^i\mathcal{F}$ can be combined into a functor between derived categories, and is usually called the derived functor.*

In general, the procedure to compute $\mathcal{R}^i(\mathcal{F})$ (and $\mathcal{L}^{-i}(\mathcal{F})$ likewise) is to use resolutions. Given $M \in A$, take its resolution $C = (0 \to M^0 = M \to M^1 \to \ldots)$, where $H^i(M) = 0$ for $i > 0$, and $H^0(C) = M$. Given a resolution C, then $\mathcal{F}(C)$ is a complex in B, and then we can compute its cohomology there.

Proposition 3. *There is always a canonical map $H^i(\mathcal{F}(C)) \to \mathcal{R}^i\mathcal{F}(M)$; moreover, it is an isomorphism if M^i are adjusted to $\mathcal{F}$. (An object M is called adjusted to $\mathcal{F}$ if $\mathcal{R}^i\mathcal{F}(M) = 0$. Of course, for left exact functors we use left resolutions.)*

An injective object is adjusted to any left exact functor. If we have *enough injectives* (i.e. for any M there is a monomorphism $M \hookrightarrow I$ into some injective object I), then any left exact functor has derived functors. Similarly we have the concept of projective objects and projective resolution. (Recall from homework that $\mathbf{QCoh}(X)$ doesn't have enough projectives, but it does have enough injectives.) One more concept: Flabby (flasque) sheaves are adjusted to Γ; by flabby we mean that for any $U \supset V, \Gamma(U, \mathscr{F}) \to \Gamma(V, \mathscr{F})$ is onto.

Recall that $\Gamma(X, \mathscr{F}) = \pi_*(\mathscr{F})$ where $\pi : X \to \mathrm{pt}$. Also recall that f_* is left exact for any $f : X \to Y$ of algebraic varieties, so we can also consider $\mathcal{R}^i f_*$. Recall also that f_* is exact if f is an affine morphism. In general (say X is separated) we can write $X = \bigcup U_i$ such that $f|_{U_i}$ is affine (e.g. U_i are affine), then compute $\mathcal{R}^i f_i \mathscr{F}$ using the Čech complex.

Proposition 4. *If f is affine, $\mathscr{F}$ is quasicoherent, then $\mathrm{H}^i f_* \mathscr{F} = \mathrm{H}^i(\mathscr{F})$.*

Proof. For separated Y, the Čech complexes agree if we use an affine covering of Y and cover X with their preimages under f. In general, can take limit over all affine coverings. □

Let X be a curve, consider $\mathscr{F} \to j_*j^*\mathscr{F} \to j_*j^*\mathscr{F}/\mathscr{F} \to 0$ for $j : U \hookrightarrow X$ of an affine set U, then we claim this is an adjusted resolution of $\mathscr{F}$ to Γ. (This links back to the beginning of the lecture.)

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