Lecture 24: Birkhoff-Grothendieck, Riemann-Roch, Serre Duality

Homework Related Stuff Remark on the 10th homework: we do have counterexamples to 5(b) if the characteristic is not 0. Consider the Drinfeld curve a.k.a. the Deligne-Lusztig variety of dimension 1, given by $x^p y - y^p x - z^{p+1} = 0$ in $\mathbb{F}_p$. $SL_2(\mathbb{F}_p)$ acts on X, (a, b, c, d) acts by sending (x, y) to $(ax+b, cx+d)$ is an isomorphism of this curve. Also, in 2b) one doesn't need the finiteness condition.

Back to Cohomology Recall that $H^*(X, \mathscr{F})$ can be computed using 1) Čech cohomology for a fixed affine covering, or 2) adjusted e.g. flabby resolution.

Remark 1. *1) is a particular case of 2). In particular, let $j : U \to X$ be an open embedding of U affine in X separated, then j_* is adjusted to Γ. Proof: j is an affine map, so $H^i(j_*\mathscr{F}) = H^i(\mathscr{F}) = 0$ for $i > 0$.*

If $X = U_1 \cup \ldots \cup U_n$, then as an example, $\bigoplus j_{i*}j_i^*\mathscr{F} \to \bigoplus j_{i_1,i_2*}j_{i_1,i_2}^*\mathscr{F} \to \ldots$ is an resolution. Another example: suppose X is an irreducible curve, $X \supset Y$, and Y is an affine open, say $X - \{x_1, \ldots, x_n\}$. If $\mathscr{F}$ has sections supported on x_i, then we have an s.e.s. $0 \to \mathscr{F} \to j_*j^*\mathscr{F} \to j_*j^*\mathscr{F}/\mathscr{F} \to 0$. Last term is flabby, since it's supported on a finite set.

Example 1. *Let's compute $H^i(\mathcal{O}_{\mathbb{P}^1}(n))$ using the 2-term complex*

$$0 \to \Gamma(\mathcal{O}_{\mathbb{P}^1}(n)) = k[X] \to \Gamma(\mathcal{O}_{\mathbb{P}^1}(n)|_{\mathbb{A}^1})/\mathcal{O}_{\mathbb{P}^1}(n)) \to 0$$

Using affine charts, one can compute the second term to be $\dfrac{k[x, x^{-1}]}{x^n k[x^{-1}]}$. The map is onto for $n \geq 0$, and the kernel consists polynomials of degree $\leq n$. Thus for $n \geq 0$, dimension of $H^0(\mathcal{O}(n)) = n+1$, and $H^1(\mathcal{O}(n)) = 0$. For the negative cases, do inverse induction using $0 \to \mathcal{O}(n-1) \to \mathcal{O}(n) \to \mathcal{O} \to 0$ or run the same argument again. In particular, when $n < 0$, H^0 is 0, and H^1 has dimension $-n-1$. So $H^0(\mathcal{O}(-1)) = H^1(\mathcal{O}(-1)) = 0$.

This yields a classification of locally free sheaves on $\mathbb{P}^1$:

Theorem 1.1 (Grothendieck-Birkhoff)**.** *A locally free coherent sheaf of rank n on $\mathbb{P}^1$ is isomorphic to $\bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(d_i)$ for a unique collection d_i.*

Proof. Uniqueness is left as an exercise; one way is to recover d_i from dimensions of $H^i(\mathcal{E}(d))$ for $i = 0, 1, d \in \mathbb{Z}$. Now let's prove existence. We use induction on rank.

Claim: $H^0(\mathcal{E}(d)) \neq 0$ for $d \gg 0$, and $= 0$ for $d \ll 0$. Proof: $\mathcal{E}$ is a quotient, i.e. $\mathcal{O}(-m)^N \twoheadrightarrow \mathcal{E}$, $\mathcal{O}(-m')^{N'} \twoheadrightarrow \mathcal{E}^\vee \implies \mathcal{E} \subset \mathcal{O}(m')^{N'}$ and so $H^0(\mathcal{E}(-d)) = 0$ for $d > m'$. For $d > m$, $\mathcal{O}(d-m)^N \twoheadrightarrow \mathcal{E}(d)$, and the first is generated by global sections. Pick d such that $\Gamma(\mathcal{E}(d)) \neq 0$ but $= 0$ for $d' < d$, and replace $\mathcal{E}$ with $\mathcal{E}(d)$, then we can assume $\Gamma(\mathcal{E}) = 0$ and $\Gamma(\mathcal{E}(d)) = 0$ for $d < 0$.

Pick some $\sigma : \mathcal{O} \to \mathcal{E}$, claim: $\mathcal{E}/\operatorname{im}(\sigma)$ has no torsion. Proof: otherwise $\mathcal{O}(D) \hookrightarrow \mathcal{E}$ for some effective divisor D, then $\Gamma(\mathcal{E}(-D)) = \Gamma(\mathcal{E}(-d)) \neq 0$ for $d = \deg(D)$, contradiction. So we have $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{E}' \to 0$, where the third is locally free. By induction, $\mathcal{E}' = \bigoplus \mathcal{O}(d_i)$.

Claim: $d_i \leq 0$. Proof: otherwise we can write $0 \to \mathcal{O}(-1) \to \mathcal{E}(-1) \to \mathcal{E}'(-1) \to 0$. $H^1(\mathcal{O}(-1)) = 0 \implies H^0(\mathcal{E}(-1)) \twoheadrightarrow H^0(\mathcal{E}'(-1))$. Suppose for some $d \geq 0$, we can write $\mathcal{E}' = \mathcal{O}(d) \oplus \ldots$, then we have $\mathcal{E}'(-1) = \mathcal{O}(d-1) \oplus \ldots$, hence $H^0(\mathcal{E}'(-1)) \neq 0 \implies H^0(\mathcal{E}(-1)) \neq 0$, contradiction.

It remains to check that the s.e.s. $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{E}' \to 0$ splits. Easier to check that the dual sequence $0 \to \mathcal{E}'^\vee \to \mathcal{E}^\vee \to \mathcal{O} \to 0$ splits. To see this, it's enough to see that $\Gamma(\mathcal{E}^\vee) \to \Gamma(\mathcal{O})$ is onto. First one is $\operatorname{Hom}(\mathcal{O}, \mathcal{E}^\vee)$, second being k. But $\mathcal{E}'^\vee$ is the sum of all $\mathcal{O}(d_i)$ where $d_i \geq 0$, so $H^1(\mathcal{E}'^\vee) = 0$, and this is the obstruction to the surjectivity using the l.e.s.

Or we can invoke a little homological algebra and just say the following: $\operatorname{Ext}^1(A, B)$ parametrizes the isomorphism classes of extensions $0 \to B \to C \to A \to 0$. Note that $\operatorname{Ext}^1(\mathcal{E}', 0) = H^1(\mathcal{E}'^\vee) = 0$. □

Here are some general facts, probably to be covered in 18.726:

1. $H^i(X, \mathscr{F}) = 0$ for $i > \dim(X)$, where $\mathscr{F}$ is an quasicoherent sheaf.

2. If X is complete and $\mathscr{F}$ coherent, then $H^i(X, \mathscr{F})$ is finite-dimensional.

The proof of these statements are beyond the scope of this course, but at least we can prove them for X of dimension 1.

Proof. We can first reduce to the case of X a smooth (eqv. normal) curve. Let $q : Y \to X$ be the normalization of X, and $\mathscr{F}$ a coherent sheaf on X. Consider $\varphi : \mathscr{F} \to q_* q^* \mathscr{F}$: the kernel and cokernel of this map are supported at singular points of X, and thus are torsion sheaves. Coherent torsion sheaves are extensions of copies of skyscraper sheaves supported at the singular points, so they have finite dimensional H^0 and higher cohomology groups vanish, so by the cohomology les it suffices to prove the corresponding statements for $q_* q^* \mathscr{F}$. Since q is an affine map, $H^i(X, q_* q^* \mathscr{F}) = H^i(q^* X, q^* \mathscr{F})$, so we reduce to the smooth case.

Now a smooth curve X admits an affine map f to the projective line $\mathbb{P}^1$, which is defined by any non-constant element of the field of rational functions when X is connected, and is finite when X is complete. We have that $H^*(X, \mathscr{F}) = H^*(\mathbb{P}^1, f_* \mathscr{F})$, so we further reduce to proving the following statements for any quasicoherent sheaf $\mathscr{F}$ on $\mathbb{P}^1$:

1. $H^i(\mathbb{P}^1, \mathscr{F}) = 0$ for $i > 1$;

2. If $\mathscr{F}$ is coherent, then H^0 and H^1 are finite dimensional.

The first statement is clear from the Cech cohomology computation, where we use the standard 2-piece affine covering. For the second one, write $\mathscr{F}$ as a sum of a locally free sheaf and a torsion sheaf. A coherent torsion sheaf on curve clearly has H^0 finite dimensional and H^1 vanishing, and the case for locally free sheaf follows from Grothendieck-Birkhoff. □

Euler Characteristic Define the *Euler characteristic* $\chi : K^0(\mathbf{Coh}(X)) \to \mathbb{Z}$ for X a complete algebraic variety. One can compute that $\chi([\mathscr{F}]) = \sum_i (-1)^i \dim H^i(\mathscr{F})$, and the l.e.s. of cohomology shows that χ is additive on short exact sequences.

Theorem 1.2 (Riemann-Roch for Curves). *Let X be irreducible complete (or smooth, for convenience's sake) curve. Then $\chi(\mathscr{F}) = \deg(\mathscr{F}) - \operatorname{rank}(\mathscr{F})(g_a - 1)$ where $g_a = \dim H^1(\mathcal{O})$.*

g_a is the arithmetic genus, which equals the geometric genus for nonsingular curves.

Proof. Enough to check on generators of $K^0(\mathbf{Coh}(X))$.

Lemma 1. *$\mathcal{O}(X)$ along with $\mathcal{O}_x$ generate the group.*

To see it implies the theorem: if $\mathscr{F} = \mathcal{O}_x$, lhs $= 1 =$ rhs. if $\mathcal{O}_X$, lhs $= 1 - g_a =$ rhs. Proof of the lemma: recall that if $\mathscr{F}$ is torsion then it is some $\bigoplus \mathcal{O}_{x_i}$. Now we do induction on rank: if $\mathscr{F}$ has rank i and torsion-free, find some $\mathscr{F}|_U = X \setminus \{x_1, \ldots, x_n\}$ that has a section $\sigma : \mathcal{O} \to \mathscr{F}$. Then it extends to $\mathcal{O}(-D) \hookrightarrow \mathscr{F}$ for $D = \sum d_i x_i$ for some $d_i > 0$, then we're done because $\mathscr{F}/\mathcal{O}(-D)$ has smaller rank, and $\mathcal{O}(-D) \equiv [\mathcal{O}] - \sum_i d_i [\mathcal{O}_{x_i}]$. □

Theorem 1.3 (Serre Duality). *If $\mathcal{E}$ is a locally free sheaf on a complete smooth (this time essential) irreducible curve, then we have a canonical isomorphism $\Gamma(\mathcal{E})^* \cong H^1(\mathcal{E}^\vee \otimes K_X)$.*

Noting that $H^1(K_X) \cong k$, and we said there's a map $H^i(\mathscr{F}) \otimes H^j(\mathscr{G}) \to H^{i+j}(\mathscr{F} \otimes \mathscr{G})$, so the pairing comes from $\mathcal{E} \otimes (\mathcal{E}^\vee \otimes K) \to K$. The proof we shall present below is based on Tate's paper [Tat68].

Proof. Recall that for $x \in X$, $\widehat{\mathcal{O}_{x,X}} \cong k[[t]]$, and the residue field is just $k((t))$, the Laurent power series. So $\widehat{\mathcal{O}_{x,X}}$ is a complete topological vector space (with Tychonoff topology), and the residue field is a linear topological vector space. Also recall an elementary duality that generalizes the usual linear duality of vector spaces, as a functor from discrete spaces to complete vector spaces, given by $V \mapsto \mathrm{Hom}(V,k)$, and the other way by $W \mapsto \mathrm{Hom}_{\mathbf{Cont}}(W,k)$. In particular, $k((t))^\vee \cong k((t))$ (the topological dual), and $k[[t]]^\vee \cong t^{-1}k[t^{-1}] \implies t^{-1}k[t^{-1}]^\vee \cong k[[t]]$ (notice this is non-canonical). Observation: we have $k((t))^\vee \cong \Omega(k((t))/k) \cong k((t))dt$ coming from the pairing $(f,\omega) \mapsto \mathrm{res}(f\omega)$.

On the other hand, we have

$$(\mathcal{E}_x \otimes_{\mathcal{O}_x,X} F_{\mathrm{res}}(\widehat{\mathcal{O}_{x,X}}))^\vee \cong (\mathcal{E}^\vee \otimes K_X) \otimes_{\mathcal{O}_{x,X}} F_{\mathrm{res}}(\widehat{\mathcal{O}_{x,X}})$$

where F_{res} denotes the residue field. Here's the overall plan of the proof: we have $Y = X \setminus \{x_1, \ldots, x_n\}$ affine. Call the left side $(\widehat{E_x}^\circ)^\vee$, and define $\widehat{E_x} = \mathcal{E}_x \otimes_{\mathcal{O}_{x,X}} \widehat{\mathcal{O}_{x,X}}$. Then cohomology of $\mathcal{E}$ is computed using the complex $\bigoplus_x \widehat{\mathcal{E}_x} \oplus \Gamma(\mathcal{E}|_Y) \to \bigoplus_x \widehat{E_x}^\circ$. We'll check that $\widehat{E_x}^\perp = (\widehat{\mathcal{E}^\vee \otimes K_X})$ and $\Gamma(\mathcal{E}|_Y)^\vee = \Gamma(\mathcal{E}^\vee \otimes K_X)$, and conclude that $(\widehat{\mathcal{E}_x}^\circ)^\vee = \widehat{\mathcal{E}^\vee \otimes K_X}^\vee$.

□

References

[Tat68] J. Tate. "Residues of differentials on curves." English. In: *Ann. Sci. Éc. Norm. Supér. (4)* 1.1 (1968), pp. 149–159. ISSN: 0012-9593.

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