Lecture 25: Proof of Serre Duality

We'll deduce the Serre duality of curves from a linear algebra observation: let $V_1, V_2 \subset V$, and define $V_1^\perp = \{\lambda \in V^* \mid \lambda(v') = 0 \ \forall v' \in V_1\}$, then $V_1^\perp, V_2^\perp \subset V^*$, then $V_1 \cap V_2 = (V^*/V_1^\perp + V_2^\perp)^*$ and $V_1^\perp \cap V_2^\perp = (V_1 + V_2)^\perp = (V/(V_1+V_2))^*$. In particular, let $C = (V_1 \oplus V_2 \to V)$ and $C' = (V_1^\perp \oplus V_2^\perp \to V^*)$, then $H^0(C') = H^1(C)^*$ and $H^1(C') = H^0(C)^*$.

Definition 1. *A Tate vector space is vector space with a topology, such that there exists a basis of neighborhoods of* 0 *consisting of vector subspaces which are commensurable.*[1]

Example 1. $V = k((t))$ *is a Tate vector space, where we consider* $t^i k[[t]]$ *as the neighborhoods of* 0.

Residue Let $x \in X$ a smooth point on a curve. $\widehat{\mathcal{O}_{x,X}} = \varinjlim_n \mathcal{O}_{x,X}/\mathfrak{m}_x^n \cong k[[t]]$, and $\widehat{\mathcal{O}_{x,X}}^\circ = F_{res}(\widehat{\mathcal{O}_{x,X}}) \cong k((t))$. Then there is a residue map $\text{Res} : \Omega_{\widehat{\mathcal{O}_{x,X}}} \otimes \widehat{\mathcal{O}_{x,X}}^\circ \to k$ by mapping $\omega = \sum a t^i dt$ to a_{-1}. This is independent of the choice of t. In char $k = 0$, the residue map is characterized by 1) $\text{Res}(df) = 0$ and 2) $\text{Res}(df/f) = 1$ for f a uniformizer. Note that suppose $f = \varphi t$ for φ invertible, then $df/f = dt/t + d\varphi/\varphi$, and the second term creates residue 0. In case of char $k = p > 0$, of course residue is no longer characterized by those two, so we need to use a stronger version of 2). A possible choice is that the residue is invariant under automorphisms of the formal Taylor series $k[[t]]$. For any scalar s in k we have an automorphism $t^n dt \mapsto s^{n+1} t^n dt$, and it's clear that the only invariant linear functional is proportional to taking the coefficient at $t^{-1}dt$.

For an algebraic group G over any field one has its Lie algebra g which acts on every G-module (as derivations). For a connected group G over a field of characteristic 0 and a G-module M, the (co)invariants of G and of g on M are the same; but this is false in characteristic p. The simplest example comes from $\mathbb{F}_p[x,y]$: the polynomial x^p is not invariant for the group GL(2) of linear transformations of the variables, but it's invariant under its Lie algebra, because derivatives of a p-th power vanish.

The group of automorphisms of $k[[t]]$ belongs to a larger class of groups; in particular, it is an infinite dimensional algebraic group (a.k.a. a group scheme of infinite type). Much of the theory goes through for this generalization. The Lie algebra is the Lie algebra of vector fields of the form $f(t)d/dt$, where $f(t) \in t^{-1}k[[t]]$. (One can consider the group $\text{Aut}(k((t)))$ whose Lie algebra is the more natural thing $\{f(t)d/dt \mid f \in k((t))\}$, but this group is even "more infinite dimensional" and there are additional technical subtleties.) Vector fields act on differential forms by Lie derivatives: $v(\omega) = L_v(\omega) = d(i_v(\omega))$, where L_v is the Lie derivative, $i_v(\omega) \in k((t))$ is the "insertion" (pairing) of the vector field and the 1-form. The condition $\text{Res}(df) = 0$ is equivalent to invariance of residue under the action of the Lie algebra, which is the same as invariance under the group if we are over a field of characteristic zero, but not in general.

Now we can define a pairing $\widehat{\mathcal{O}_{x,X}} \times \left(\widehat{\mathcal{O}_{x,X}}^\circ \otimes \Omega\right) \to k$ that sends (f, ω) to $\text{Res}(f\omega)$. Under this we have $(\widehat{\mathcal{O}_{x,X}}^\circ \otimes \Omega) \cong (\widehat{\mathcal{O}_{x,X}}^\circ)^\vee$ as dual topological spaces, where the dual basis for t^i on the left is $t^{-i-1}dt$ on the right. (Check that left equals $k[t^{-1}] \oplus k[[t]]$, and $k[t^{-1}]^\vee = k[[t]]dt$ and $k[[t]]^\vee = t^{-1}k[t^{-1}]dt$.) So if we take the non-localized version $(\widehat{\mathcal{O}_{x,X}} \otimes \Omega)^\perp \cong \widehat{\mathcal{O}_{x,X}}$, then again we can do calculation: $\sum_{i=-N}^{\infty} a_i t^i dt$ pairing with $\sum_{i=0}^{\infty} b_i t^i$ yield 0 for all b_i iff $a_i = 0$ for $i < 0$.

Lemma 1. *Suppose* X *is a complete smooth curve,* $\omega \in \Gamma(U, \Omega)$, U *is a nontrivial open subset, then* $\sum_{x \in X \setminus U} \text{Res}_{x_i} \omega = 0$.

Sketch of Proof. (See [Tat68] for another proof.) If $X = \mathbb{P}^1$, then it is an explicit computation, as ω is a linear combination of $\dfrac{dz}{(z-a)^n}$. For general X, reduce to $X = \mathbb{P}^1$ as follows: Find a finite separable map $X \xrightarrow{\varphi} \mathbb{P}^1$, $\omega = f \circ \varphi^*(\theta)$, $f \in R(X)$, $R(X)/R(\mathbb{P}^1)$ is a finite extension, and let $\overline{f} = \text{Tr}(f) \in R(\mathbb{P}^1)$ under

[1] We say V_1 and V_2 are *commensurable* if $V_1/(V_1 \cap V_2)$ has finite dimension.

this extension. Then one can check that $\operatorname{Res}_x \overline{f}\theta = \sum_{x_i \mapsto x} \operatorname{Res}_{x_i}(\omega)$ for any $x \in \mathbb{P}^1$. As a corollary, we have $\sum_{x\in X} \operatorname{Res}(\omega) = \sum_{y \in \mathbb{P}^1} \operatorname{Res}(\overline{f}\theta) = 0$. □

Proof for Serre duality for curves. Let $\mathcal{E}$ be locally free, $Y = X \setminus \{x_1, \dots, x_n\}$ be affine, and $j : Y \hookrightarrow X$. $\widehat{\mathcal{E}_x} = \varinjlim \mathcal{E}_x/\mathfrak{m}_x^n = \mathcal{E}_x \otimes_{\mathcal{O}_{x,X}} \widehat{\mathcal{O}_{x,X}} \cong k[[t]]^r$ and $\widehat{\mathcal{E}_x}^\circ = \widehat{\mathcal{E}_x} \otimes_{\widehat{\mathcal{O}_{x,X}}} \widehat{\mathcal{O}_{x,X}} \cong k((t))^r$ where r is the rank of $\mathcal{E}$. We claim that $H^*(X, \mathcal{E})$ is computed by the complex

$$\Gamma(\mathcal{E}|_Y) \oplus \bigoplus_i \widehat{\mathcal{E}_{x_i}} \to \bigoplus_i \widehat{\mathcal{E}_{x_i}}^\circ$$

One can check its cohomology is the same as the cohomology of the complex

$$\Gamma(\mathcal{E}|_Y) \to \bigoplus_i \widehat{\mathcal{E}_{x_i}}^\circ / \widehat{\mathcal{E}_{x_i}}$$

But the right hand side is just the global section of $j_*j^*\mathcal{E}/\mathcal{E}$. Note that rhs at x is $\mathcal{E}_x \otimes_{\mathcal{O}_{x,X}} \left(\frac{\widehat{\mathcal{O}_{x,X}}^\circ}{\widehat{\mathcal{O}_{x,X}}} \right)$, and this is the stalk of $j_*j^*\mathcal{E}/\mathcal{E}$ at x. (Some more explanation: $\frac{\widehat{\mathcal{O}_{x,X}}^\circ}{\widehat{\mathcal{O}_{x,X}}} = F_{\text{res}}(\mathcal{O}_{x,X})/\mathcal{O}_{x,X} = k[U - x]/k[U]$ where U is an affine neighborhood of x. This is a module where $\mathfrak{m}_x$ acts by a local map where neither localizing by elements in $\mathfrak{m}_x$ nor replacing $\mathcal{O}_{x,X}$ by $\widehat{\mathcal{O}_{x,X}}$ affects it.)

Now set $V = \bigoplus_i \widehat{\mathcal{E}_{x_i}}^\circ \supset V_1 = \Gamma(\mathcal{E}|_Y), V_2 = \bigoplus \widehat{\mathcal{E}_{x_i}}$. Then we have the topological dual $V^\vee = \bigoplus_i (\widehat{\mathcal{E}^\vee \otimes \Omega})^\circ_{x_i}$; set $V_1' = \Gamma(\Omega \otimes \mathcal{E}^\vee|_Y), V_2' = \bigoplus \widehat{\Omega \otimes \mathcal{E}^\vee_{x_i}}$. By the linear algebra discussed above, it remains to check $V_1^\perp = V_1'$ and $V_2^\perp = V_2'$. $V_2^\perp = V_2'$ reduces to $k[[t]]^\perp \cong k[[t]]dt$. We also have $V_1' \subset V_1^\perp$, which follows from $\sum \operatorname{Res}_{x_i}\omega = 0$ (the lemma above), and it remains to see $V_1' = V_1^\perp$. Notice that $V_1' = V_1^\perp \Leftrightarrow \dim(H^i(\mathcal{E}^\vee \otimes \Omega)) = \dim(H^{1-i}(\mathcal{E}))$ by what we know.

We want to check that $V_1^\perp/V_1'$ is finite dimensional. $V_1 \subset V = k[[t]]]^r$, and as a subspace it is discrete and cocompact, i.e. has a compact complement. Discrete follows from H^0 being finite dimensional, and cocompact follows from H^1 being finite dimensional. Now, V_1 is discrete implies V_1^* is compact (complete) which implies $V_1^\perp$ is cocompact, and V_1 cocompact implies $V_1^\perp = (V/V_1)^*$ is discrete since V/V_1 is compact. Now in general, for discrete cocompact subspaces $U \subset W$ of V, one can check that the quotient W/U is discrete compact and finite dimensional.

Now we have that $V_1^\perp$ contains V_1' with finite codimension (thus the quotient $k[Y]$-module $V_1^\perp/V_1'$ is supported at finitely many points $y_1, \dots, y_m$), we can consider it as a subspace of $K(\Omega \otimes \mathcal{E}^\vee|_Y)$, the space of rational sections of $\Omega \otimes \mathcal{E}^\vee|_Y$.

From here there are two ways to proceed: on one hand, we can replace Y by $Y' = Y \setminus \{y_1, \dots, y_m\}$. Then $\Gamma(\mathcal{E}|_{Y'})^\perp = \Gamma(\mathcal{E}|_Y)^\perp{}_{(f_1,\dots,f_m)}$ where localization by f_i correspond to removing y_i (observe that if $s \in \Gamma(\mathcal{E}|_{Y'})^\perp \subset K(\Omega \otimes \mathcal{E}^\vee|_Y)$ and s is regular at each y_i, then $s \in \Gamma(\mathcal{E}|_Y)$), and we still get rational sections that may be singular at y_i; on the other hand, $\Gamma(\Omega \otimes \mathcal{E}^\vee|_{Y'})$ consists of rational sections of $\Omega \otimes \mathcal{E}^\vee$ on Y that may be singular on y_i, so we have $V_1^\perp = V_1'$ for Y'. On the other hand, we can directly check $V_1^\perp \supset V_1'$: suppose s is a rational section in $V_1^\perp$, and has singularities $y_1, \dots, y_m$. Then since Y is affine, one can find a section s' of $\mathcal{E}$ such that (s, s'), which is a section of Ω, is regular at y_i for $i > 1$, but $\operatorname{Res}_{y_1}(s, s') \neq 0$. Then we see that s cannot be orthogonal to s'. □

Now we state some standard corollaries.

Corollary 1. *Define the arithmetic genus* $g_a = \dim(H^1(\mathcal{O}))$, *and the geometric genus* $g_m = \dim(G(K_X))$. *Then apply Serre duality to* $\mathcal{E} = \mathcal{O}$ *to get* $g_a = g_m$.

Corollary 2. *Riemann-Roch implies* $\dim(\Gamma(\mathcal{E})) - \dim(\Gamma(K \otimes \mathcal{E}^*)) = \deg(\mathcal{E}) + \operatorname{rank}(\mathcal{E})(1 - g)$. *This is Riemann's form of the theorem.*

Corollary 3. $\deg(K) = 2g - 2$.

Proof. $\chi(\mathcal{O}) = -\chi(K)$ by Serre duality. $\deg(K) = \chi(K) + g - 1 = 2g - 2$. □

The statement of the Serre duality generalizes: let X be a smooth complete (irreducible) variety of dimension n, and let $\mathcal{E}$ be a locally free sheaf, then there is a duality $H^{n-i}(\mathcal{E}^\vee \otimes K) \cong H^i(\mathcal{E})^*$. It can also be generalizred to not locally free sheaves and non-smooth varieties (best described using derived categories).

For instance, let X be a smooth affine curve, and $\mathscr{F}$ a torsion sheaf. Then there exists a canonical isomorphism $\Gamma(\mathscr{F})^* \cong \mathrm{Ext}^1(\mathcal{F}, K_X)$. Suppose X is smooth of dimension n, and $\mathscr{F}$ torsion is supported at a 0-dimensional set, then $\Gamma(\mathscr{F})^* \cong \mathrm{Ext}^m(\mathscr{F}, K_X)$. Generalizations of Riemann-Roch include the Hirzebruch-Riemann-Roch theorem and the Grothendieck-Riemann-Roch theorem.

Let X complete, $\mathscr{F}$ coherent sheaf, $\chi(\mathscr{F})$ is a topological invariant of $\mathscr{F}$, i.e. one can give a formula for $\chi(\mathscr{F})$ in terms of topological invariants of $\mathscr{F}$ and that of the tangent bundle of X. For instance, suppose X is locally free and is over $\mathbb{C}$, then it corresponds to a vector bundle, and has Chern classes. Then $\chi(\mathscr{F})$ is expressed via the Chern classes. In particular, it's constant in families. Even more generally, recall that the global section functor is the same as direct image of the map to a point, and cohomology are the higher direct images. So if we replace $X \to \mathrm{pt}$ to an arbitrary map $X \to Y$, we get Grothendieck's version of Riemann-Roch.

A major theme of AG is the question of how to reconstruct topological invariants of $X(\mathbb{C})_{cl}$ (classical) from AG data. This of course can also generalize to other fields. There are two approaches: the de Rham approach (using differentials, e.g. if X is an affine smooth variety, then X's regular cohomology can be computed using its algebraic de Rham complex $k[X] \xrightarrow{d} \Gamma(\Omega^1 X) \xrightarrow{d} \Gamma(\Omega^2 X) \to \dots$ where $\Omega^i X = \bigwedge^i \Omega X$), and the etale approach (related to counting of $X(\mathbb{F}_q)$ and the Weil conjectures).

References

[Tat68] J. Tate. "Residues of differentials on curves." English. In: *Ann. Sci. Éc. Norm. Supér. (4)* 1.1 (1968), pp. 149–159. ISSN: 0012-9593.

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