18.726 Algebraic Geometry
Spring 2009

18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Category theory (updated 8 Feb 09)

We're going to use the language of category theory freely. Fortunately, it's easy to learn because it corresponds naturally to the way you (hopefully) already think about mathematical objects. (I could give a reference, but in fact you should be fine just looking these things up in Wikipedia.)

1 Warning: set-theoretic difficulties

Category theory is a bit tricky because it tries to deal with objects like "the ??? of all sets", or all rings, or whatnot. Russell's paradox shows that there is in fact no *set* of all sets. Namely, if there were, it would have a subset consisting of those sets U for which $U \notin U$. But that would then be a set V, and if $V \in V$ then $V \notin V$ and vice versa.

The way around this is to tamper with the axioms of set theory slightly, by introducing the notion of a *class*. A class is something which behaves just like a set whose members are sets, except that there is no power axiom; there is not guaranteed to be a class consisting of all subclasses of a given class. Unless, that is, your class consists just of the elements of some actual set. (You might then ask what kind of object is the ??? of all classes. Never mind that for now.) We also assume there is a class of all sets, called the *universe*.

Except for the power axiom, you may perform operations on classes like you do with sets. For instance, given a class C and a logical statement P depending on a single set, you can form the subclass of C consisting of all elements for which P is true. You can also form Cartesian products indexed by sets, although I'll hardly ever do this except for *finite* products. (There is also an axiom of choice at the class level.)

A class is *small* if its elements are in bijection with some set.

2 Categories, and examples

A *category* $\mathcal{C}$ consists of the following data.

- A class of *objects*, denoted $\mathrm{Obj}(\mathcal{C})$.
- For each ordered pair of objects (X, Y), a set of *morphisms*, denoted $\mathrm{Hom}(X, Y)$. (You may think of this as an element of the Cartesian product of two copies of $\mathcal{C}$ and one copy of the universe.)
- For each ordered triple of objects (X, Y, Z), a function $\circ : \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$, called *composition*, which satisfies the following properties.
  - The *associative law*: given an ordered quadruple of objects (X, Y, Z, W), the two ways to compose $\mathrm{Hom}(Z, W) \times \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y)$ to $\mathrm{Hom}(X, W)$ give the same answer.

– The *identity law*: for each object X, there must exist a morphism $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ which is an identity under composition on either side. Note that id_X is forced to be unique by this condition.

(I have a habit of calling morphisms "arrows" because they are usually pictorially represented as such.)

This definition is meant to capture many, if not all, basic types of structured mathematical objects. Examples:

- The category of sets, denoted Set, where $\mathrm{Hom}(X, Y)$ is all functions from X to Y.
- The category of topological spaces, denoted Top, where $\mathrm{Hom}(X, Y)$ is all *continuous* functions from X to Y.
- The category of (commutative, unital) rings, denoted Ring, where $\mathrm{Hom}(X, Y)$ is all *ring homomorphisms* from X to Y.
- The category of topological rings, denoted TopRing, where $\mathrm{Hom}(X, Y)$ is all *continuous ring homomorphisms* from X to Y.
- The category of modules over a fixed ring R, denoted Mod_R, where $\mathrm{Hom}(X, Y)$ is all *R-module homomorphisms* from X to Y.

And so forth. I'll leave to your imagination the definitions of some more categories for which I might need names later: Ab (abelian groups), Grp (groups), TopGp (topological groups).

However, there are other things that can be viewed as categories. An important example: given any partially ordered set S, make a category in which the objects are the elements of S, and there is exactly one morphism from X to Y if $X \leq Y$ and none otherwise.

Important special case of the previous one: given a topological space S, we can make a category in which the objects are the open subsets of S, and the morphisms are the inclusions of one open subset into another.

Another example comes from algebra. Given a group, we can make a category with only one object X, in which $\mathrm{Hom}(X, X)$ is the group and the composition law is the group operation.

Here's a more interesting example along the lines of the previous one. Given a topological space S, make a category in which the objects are the *points* of S, and the morphisms from X to Y are the continuous functions $f : [0, 1] \to S$ with $f(0) = X$ and $f(1) = Y$. Define the composition $g \circ f$ for $g \in \mathrm{Hom}(Y, Z)$ and $f \in \mathrm{Hom}(X, Y)$ to be the function $h : [0, 1] \to S$ with

$$h(x) = \begin{cases} f(2x) & x \in [0, 1/2] \\ g(2x - 1) & x \in [1/2, 1]. \end{cases}$$

This is a special case of turning a *groupoid* (something which is like a group except that objects can only be composed if they satisfy a matching condition) into a category. This example comes from the *fundamental groupoid* of a topological space.

3 Interlude: "is" versus "does"

The rigorous formulation of category theory exposes a dark secret of mathematics: objects in a category are rarely ever *equal*. For instance, we all think we agree on what the ring $\mathbb{Z}$ is, but if we all sat down and wrote down set-theoretic definitions, probably no two of them would exactly match. The point is that we conceive of $\mathbb{Z}$, and of most mathematical objects in general, not in terms of what they literally *are* as sets, but by how they *work*, and in particular how they relate to other mathematical objects.

The solution for this suggested by category theory is to characterize interesting mathematical objects using *universal properties*. For instance, the ring $\mathbb{Z}$ is characterized by the fact that it is an *initial object* in the category of rings: for every ring Y, there is a unique morphism from $\mathbb{Z}$ to Y. Any two objects with this property are *uniquely* isomorphic.

Here are a few other "arrow-theoretic" properties that can be used for this purposes. I'll talk more about universal properties later.

- $Y \in \mathrm{Obj}(\mathcal{C})$ is a *final object* in $\mathcal{C}$ if for any $X \in \mathrm{Obj}(\mathcal{C})$, there is a unique morphism from X to Y. An object which is both initial and final is a *terminal object*.

- A morphism $f \in \mathrm{Hom}(X, Y)$ is a *monomorphism* if for any $g, h \in \mathrm{Hom}(W, X)$, if $f \circ g = f \circ h$, then $g = h$. In the category of sets (and many other examples), f is a monomorphism if and only if f is injective.

- A morphism $f \in \mathrm{Hom}(X, Y)$ is an *epimorphism* if for any $g, h \in \mathrm{Hom}(Y, Z)$, if $g \circ f = h \circ f$, then $g = h$. In the category of sets (and many other examples), f is an epimorphism if and only if f is surjective. But beware of surprises: for example, the morphism $\mathbb{Z} \to \mathbb{Q}$ of rings is an epimorphism (and also a monomorphism).

- A morphism $f \in \mathrm{Hom}(X, Y)$ is an *isomorphism* if it has a two-sided inverse. This implies that it is a monomorphism and an epimorphism, but not conversely (see previous example).

4 Functors and natural transformations

Functors can be thought of as "functions between categories". A *covariant functor* from a category $\mathcal{C}_1$ to a category $\mathcal{C}_2$ consists of:

- A function F from $\mathrm{Obj}(\mathcal{C}_1)$ to $\mathrm{Obj}(\mathcal{C}_2)$.

- For each pair (X, Y) of $\mathrm{Obj}(\mathcal{C}_1)$, a function $F_{X,Y} : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$, such that F commutes with composition and F carries id_X to $\mathrm{id}_{F(X)}$.

A *contravariant functor* works the same way except that $F_{X,Y}$ carries $\mathrm{Hom}(X, Y)$ to $\mathrm{Hom}(F(Y), F(X))$, that is, it reverses the sense of the morphisms. You can turn it into a covariant functor by replacing one of the two categories with its *opposite category*, in which all morphisms are reversed; for simplicity, let us just talk about covariant functors for the moment.

This point is actualized by the notion of a *natural transformation*. Given two functors F_1, F_2 from $\mathcal{C}_1$ to $\mathcal{C}_2$, a *natural transformation* of F_1 to F_2 consists of, for each $X \in \mathrm{Obj}(\mathcal{C}_1)$, a morphism $\phi_X : F_1(X) \to F_2(X)$ such that for every morphism $f \in \mathrm{Hom}(X, Y)$, the diagram

$$\begin{array}{ccc} F_1(X) & \xrightarrow{F_1(f)} & F_1(Y) \\ \downarrow{\scriptstyle \phi_X} & & \downarrow{\scriptstyle \phi_Y} \\ F_2(X) & \xrightarrow{F_2(f)} & F_2(Y) \end{array}$$

is commutative (that is, if you trace around both ways you get the same answer). Natural transformations can be composed; one with an inverse (equivalently, in which the morphisms ϕ_X are all isomorphisms) is called a *natural isomorphism*. For instance, the functors taking ordered triples (M_1, M_2, M_3) of modules over a ring R to

$$(M_1 \otimes_R M_2) \otimes_R M_3 \qquad \text{and} \qquad M_1 \otimes_R (M_2 \otimes_R M_3)$$

are naturally isomorphic.

5 Other properties of functors

A functor is *faithful* if the maps $F_{X,Y}$ are injective. Typical examples of these are "forgetful" functors, in which you start with a category of objects carrying a lot of structure, and the functor strips off some structure. E.g., take groups to their underlying sets, or take rings to their additive groups, or take topological groups to their underlying topological spaces.

The analogues of injectivity and surjectivity for functors are:

- A functor is *fully faithful* if the maps $F_{X,Y}$ are *bijective*. A typical example is the inclusion of a *full subcategory* (i.e., take some of the objects, and all of the morphisms between the chosen objects).

- A functor is *essentially surjective* if every object in $\mathcal{C}_2$ is isomorphic to an object of the form $F(X)$ for some $X \in \mathrm{Obj}(\mathcal{C}_1)$.

- A functor is an *equivalence of categories* if it is fully faithful and essentially surjective. This is equivalent to the existence of a *quasi-inverse* functor, i.e., one for which the compositions in both directions are naturally isomorphic to the relevant identities.

A typical example from last semester: take the category of affine algebraic varieties over an algebraically closed field k. The functor Ω computing regular functions is an equivalence between this category and (the opposite category of) finitely generated k-algebras which are *reduced* (have no nilpotent elements). One of the goals of schemes is to set up a similar equivalence between some sort of geometric objects and the category of *all* commutative unital rings.

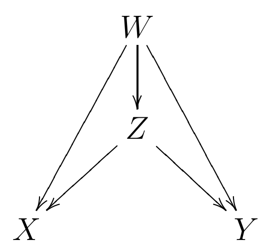
6 Representable functors, Yoneda's lemma, and universal properties

An individual object in a category casts a sort of shadow on the entire category, via the notion of representable functors. For a fixed object X in a category $\mathcal{C}$, let h_X be the functor from $\mathcal{C}$ to Set such that $h_X(Y) = \mathrm{Hom}(X, Y)$, and the image of $f \in \mathrm{Hom}(Y, Z)$ under h_X carries $\mathrm{Hom}(X, Y)$ to $\mathrm{Hom}(Y, Z)$ via postcomposition with f.

It turns out that any natural transformation from h_X to any other functor $F : \mathcal{C} \to \mathrm{Set}$ is determined by specifying the image of the special element id_X of $\mathrm{Hom}(X, X) = h_X(X)$, and conversely any such choice induces a natural transformation from h_X to F. This is *Yoneda's lemma*; proof is left as an (easy) homework problem.

An arbitrary functor $F : \mathcal{C} \to \mathrm{Set}$ is *representable* if it is naturally isomorphic to h_X for some X. By Yoneda's lemma, if X and Y represent the same functor, then they are isomorphic in a "natural" way (i.e., one compatible with the action of the functor).

In practice, this is usually interpreted as saying that an object of a category determined by a *universal mapping property* is *unique up to unique isomorphism* (or *up to natural isomorphism*). Here is an example of this which will help explain why categorical thinking is so helpful when dealing with schemes. For objects X, Y in a category $\mathcal{C}$, an *(absolute) product* of X and Y is an object Z equipped with maps $Z \to X$ and $Z \to Y$, with the following universal mapping property. Given any object W and morphisms $W \to X$ and $W \to Y$, there must be a *unique* morphism $W \to Z$ such that the diagram



commutes. The product is unique in the sense that if Z' is an other object equipped with morphisms $Z' \to X$ and $Z' \to Y$ satisfying the mapping property, there is a unique isomorphism $Z \to Z'$ making everything commute.

In any "normal" category, in which objects are sets equipped with some extra structure (e.g., groups, topological groups), products exist and can be written as Cartesian products with some appropriate extra structure. But in general, products need not exists, and even if they do they might look weird. Case in point: suppose we tried to make a theory of abstract algebraic varieties over the non-algebraically closed field $\mathbb{Q}$, in which the points are Galois orbits of points over $\overline{\mathbb{Q}}$. (This is close to what will happen with schemes, except that there will be some more points.) Then in the affine line, we have a variety consisting of the single orbit $\{i, -i\}$. The product of this with itself will then consist of the *two* orbits $\{(i, i), (-i, -i)\}$ and $\{(i, -i), (-i, i)\}$.

7 Limits and colimits

The universal mapping properties we will consider can all be wrapped into the following framework. Let $\mathcal{C}, \mathcal{D}$ be two categories. A *diagram* on $\mathcal{C}$ of type $\mathcal{D}$ is just a functor from $\mathcal{D}$ to $\mathcal{C}$.

Fix a diagram $F : \mathcal{D} \to \mathcal{C}$. Let $\mathcal{D}'$ be the category formed from $\mathcal{D}$ by adding one extra object I with a unique morphism to every object in $\mathcal{D}'$ (and the obvious composition law). Now look at extensions of F to functors $\mathcal{D}' \to \mathcal{C}$; that is, you have to add one object X of $\mathcal{C}$ and maps $X \to F(Y)$ for each $Y \in \mathcal{D}$ which commute with the maps coming from the diagram. A *limit* of F is a universal set of such data, i.e., any other extension factors uniquely through this one. My example of a product is the case where $\mathcal{D}$ consists of two objects and no morphisms.

Define *colimits* as limits in the opposite category. For example, the co-analogue of the product is the *coproduct*. In Set, the product is the Cartesian product, while the coproduct is the disjoint union.

Important special case: a *directed set* is a partially ordered set in which any two elements have a common upper bound. (I.e., for any x, y, there is some z with $x \leq z$, $y \leq z$. A diagram from a directed set into some category $\mathcal{C}$ is called a *direct system*; a colimit of a direct system is called a *direct limit*, or an *inductive limit*, in $\mathcal{C}$. (It should be called a *direct/inductive colimit*. Sorry about that.) For example, take the natural numbers under divisibility; then the direct limit of the abelian groups $\frac{1}{n}\mathbb{Z}$ is the group $\mathbb{Q}$.

A diagram from the opposite of a directed set into some category $\mathcal{C}$ is called an *inverse system*; a colimit of an inverse system is called an *inverse limit* (or *projective limit*). For example, view the nonnegative integers as a partially ordered set using the reverse of the usual ordering. Then for any ring R, the inverse limit of the rings $R[x]/(x^n)$ is the ring $R[\![x]\!]$ of formal power series. (A similar example is the p-adic numbers.)

8 Adjoint functors

One other notion that comes up a lot is that of an *adjoint pair of functors*, which you might like to think of as category-theoretic analogues of a linear operator and its transpose.

Let $\mathcal{C}, \mathcal{D}$ be categories. A pair of functors $F^* : \mathcal{C} \to \mathcal{D}$ and $F_* : \mathcal{D} \to \mathcal{C}$ form an *adjoint pair* if we can form bijections

$$\mathrm{Hom}_{\mathcal{C}}(F^*X, Y) \to \mathrm{Hom}_{\mathcal{D}}(X, F_*Y)$$

which are functorial in X and Y (imagine the diagrams yourself). In this relationship, F^* is the *left adjoint* and F_* is the *right adjoint*.

The notation was chosen because the adjoint pairs we will use correspond to operations of *promotion* and *demotion* between two categories, one of which has more structured objects than the other. Here is a typical example. Let $F_* : \mathrm{Ab} \to \mathrm{Set}$ be the forgetful functor on abelian groups. Let $F^* : \mathrm{Set} \to \mathrm{Ab}$ be the functor carrying a set S to the *free abelian group* generated by S. Then F^* and F_* form an adjoint pair.

Another important example for us: let $R \to S$ be a homomorphism of rings. Let $F^* : \mathrm{Mod}_R \to \mathrm{Mod}_S$ be the functor $M \mapsto M \otimes_R S$. Let $F_* : \mathrm{Mod}_S \to \mathrm{Mod}_R$ be the functor given by restriction of scalars from S to R. Then F^* and F_* form an adjoint pair.

We can of course compose F^* and F_* both ways, and we don't in general get the identity, or even something naturally isomorphic to the identity. We do get something interesting, though. The identity map on F^*X corresponds to a morphism $X \to F_*F^*X$, while the identity map on F_*Y corresponds to a morphism $F^*F_*Y \to Y$. These morphisms are called *adjunction morphisms*. For example, in the previous example, for X an R-module, $X \to F_*F^*X = X \otimes_R S$ is the map $x \mapsto x \otimes 1$. For Y an S-module, $Y \otimes_R S \to Y$ is the map $\sum_i y_i \otimes s_i \mapsto \sum_i y_i s_i$.