18.726 Algebraic Geometry
Spring 2009

18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Flat morphisms and descent (updated 11 Mar 09)

Hartshorne only treats flatness after cohomology (so see III.9) and doesn't talk about descent at all. The EGA reference for flatness is EGA IV, part 2, §2. I'm not sure if descent is discussed at all in EGA, so I gave references to SGA 1 instead.

1 Flat sheaves and flat morphisms

Let $f : Y \to X$ be a morphism and let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_Y$-module. We say $\mathcal{F}$ is *flat relative to* f if for each point $y \in Y$ with $f(y) = x$, if we use the map $f^\sharp : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ to view $\mathcal{F}$ as a $\mathcal{O}_{X,x}$-module, then that module is flat in the usual sense. (The usual sense is that an R-module M is *flat* if tensoring with it is exact, not just right exact.) If this holds at a particular y, we say $\mathcal{F}$ is *flat at* y *relative to* f.

Two special cases:

- If $Y = X$, we say that $\mathcal{F}$ is a *flat* $\mathcal{O}_X$*-module*; it is equivalent to saying that tensoring with $\mathcal{F}$ is an exact functor on quasicoherent $\mathcal{O}_X$-modules. For instance, any locally free $\mathcal{O}_X$-module is flat.
- If $\mathcal{F} = \mathcal{O}_Y$, we say that f is a *flat morphism*. For example, any open immersion is flat.

Note that if $\mathcal{F}$ is a flat $\mathcal{O}_Y$-module and f is a flat morphism, then $\mathcal{F}$ is flat relative to f. Note that also that flatness is *local on the source*, not just on the target, and stable under base change.

Lemma. *Let* $X = \mathrm{Spec}(R)$ *be an affine scheme, and let* M *be an* R*-module. Then* $\tilde{M}$ *is a flat* $\mathcal{O}_X$*-module if and only if* M *is a flat* R*-module.*

Proof. This should be a familiar fact from commutative algebra: M is flat over R if and only if $M_\mathfrak{p}$ is flat over $R_\mathfrak{p}$ for each prime ideal $\mathfrak{p}$. For completeness, I include the proof here.

Suppose first that M is flat. Let $\mathfrak{p}$ be an ideal and let $N \to P$ be an injection of $R_\mathfrak{p}$-modules. We may then view N, P as R-modules and identify

$$M_\mathfrak{p} \otimes_R N = M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N$$

and similarly for P. Since localization is flat, $R_\mathfrak{p}$ is a flat R-algebra, so $M_\mathfrak{p}$ is flat not just over $R_\mathfrak{p}$ but also over R. Hence $M_\mathfrak{p} \otimes N \to M_\mathfrak{p} \otimes P$ is injective, so $M_\mathfrak{p}$ is flat over $R_\mathfrak{p}$.

Suppose next that $M_\mathfrak{p}$ is flat over $R_\mathfrak{p}$ for each $\mathfrak{p}$. If $N \to P$ is an injection of R-modules, we must check that $M \otimes N \to M \otimes P$ is still injective. Localizing gives $M_\mathfrak{p} \otimes N_\mathfrak{p} \to M_\mathfrak{p} \otimes P_\mathfrak{p}$ (since localization commutes with tensor product), which is injective because $M_\mathfrak{p}$ is flat. $\square$

Corollary. *Let* $A \to B$ *be a homomorphism of rings. Then* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is flat if and only if* B *is flat as an* A*-module.*

Proof. The statement that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is flat says that for each $\mathfrak{q} \in \mathrm{Spec}(B)$ mapping to $\mathfrak{p} \in \mathrm{Spec}(A)$, the morphism $A_\mathfrak{p} \to B_\mathfrak{q}$ is flat. This follows from $A \to B$ being flat because the localization $B_\mathfrak{p} \to B_\mathfrak{q}$ is flat. Conversely, suppose that this holds. Let $N \hookrightarrow P$ be an injection of A-modules. Then for each prime ideal $\mathfrak{p}$ of A, we may view $B_\mathfrak{p} \otimes_A N \to B_\mathfrak{p} \otimes_A P$ as a morphism of $B_\mathfrak{p}$-modules. For each prime ideal $\mathfrak{q}$ of B over $\mathfrak{p}$, tensoring with $B_\mathfrak{q}$ over $B_\mathfrak{p}$ simply gives $B_\mathfrak{q} \otimes_A N \to B_\mathfrak{q} \otimes_A P$. This is injective because $A \to A_\mathfrak{p}$ is flat always and $A_\mathfrak{p} \to B_\mathfrak{q}$ is flat by hypothesis.

Applying the previous lemma over $B_\mathfrak{p}$, we may now deduce that $B_\mathfrak{p} \otimes_A N \to B_\mathfrak{p} \otimes_A P$ is injective. That is, $B_\mathfrak{p}$ is flat over A, or equivalently over $A_\mathfrak{p}$. Applying the previous lemma over A, we deduce that B is flat over A. □

The notion of flatness, while useful (especially when we study cohomology), is geometrically somewhat mysterious. For projective morphisms, one can give a geometric interpretation in terms of *Hilbert polynomials*; more on that later. In the interim, you may wish to chew on the following examples. (See Eisenbud-Harris II.3.4 for more examples.)

Let k be an algebraically closed field. The morphism

$$\mathrm{Spec}\, k[x,t]/(x^2 - t) \to \mathrm{Spec}\, k[t]$$

is flat. If the characteristic of k is not 2, then the fibres above points $t \neq 0$ are pairs of distinct points whereas the fibre above $t = 0$ is the doubled origin in $\mathrm{Spec}\, k[x]$.

The morphism

$$\mathrm{Spec}\, k[x,t]/(x^2 - t^2) \to \mathrm{Spec}\, k[t]$$

is also flat, but the source is not normal. If we replace the source by its normalization, we get two copies of the affine line mapping to one affine line, and this is *also* flat.

Hartshorne gives the example of the family of cubic curves in $\mathbb{A}^3$ given as parametric equations in u by

$$x = u^2 - 1, y = u^3 - u, z = tu.$$

If we eliminate u and make sure the result is flat over $\mathrm{Spec}\, k[t]$, we get

$$\mathrm{Spec}\, k[x,y,z,t]/(t^2(x+1) - z^2, tx(x+1) - yz, xz - ty, y^2 - x^2(x+1)) \to \mathrm{Spec}\, k[t].$$

The fibre over $t = 0$ is supported on the plane curve $y^2 = x^2(x+1), z = 0$ but is not a subscheme of the plane $z = 0$ in $\mathrm{Spec}\, k[x,y,z]$: the local ring at the origin contains the nonzero nilpotent element z.

Here are some deep results about flatness. For this one, see EGA 4, part 2, Théoreme 2.4.6

Theorem 1. *Let $f : X \to Y$ be a morphism which is flat and locally of finite presentation. Then f is* universally open*, i.e., any base change of f is an open map (the image of any open set is open) on topological spaces.*

For this one, see SGA 1, Exposé IV, Théorème 6.10 or EGA 4, part 3, 11.1.1.

Theorem 2. *Let $f : Y \to X$ be a morphism of finite type, with X locally noetherian, and let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_Y$-module. The set of $y \in Y$ at which $\mathcal{F}$ is flat relative to f is an* open *subset of U.*

2 Faithfully flat morphisms and descent

A morphism which is both flat and surjective is *faithfully flat*. For instance, if $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a morphism of affine schemes, then this morphism is faithfully flat if and only if B is faithfully flat in the usual sense, i.e., B is flat over A, and for any A-module M, the map $M \to M \otimes_A B$ of A-modules is injective.

Faithfully flat morphisms are important because of their role in *descent*, the process of "undoing" a base change. Here is a typical example.

Let $f : Y \to X$ be a morphism. Let $\pi_1, \pi_2 : Y \times_X Y \to Y$ be the canonical projections. The category of *descent data for quasicoherent sheaves* relative to f is defined as follows. A descent datum is a quasicoherent $\mathcal{O}_Y$-module $\mathcal{F}$ equipped with an isomorphism $\psi : \pi_1^*\mathcal{F} \to \pi_2^*\mathcal{F}$, satisfying the following *cocycle condition*. Let $\pi_1, \pi_2, \pi_3 : Y \times_X Y \times_X Y \to Y$ be the canonical projections. Use ψ first to identify $\pi_1^*\mathcal{F}$ with $\pi_2^*\mathcal{F}$, then $\pi_2^*\mathcal{F}$ with $\pi_3^*\mathcal{F}$. The resulting isomorphism $\pi_1^*\mathcal{F} \to \pi_3^*\mathcal{F}$ must coincide with the one obtained directly by applying ψ to the first and third factors.

A morphism of two descent data is a morphism $\mathcal{F} \to \mathcal{G}$ of the underlying $\mathcal{O}_Y$-modules, such that the induced morphisms $\pi_1^*\mathcal{F} \to \pi_1^*\mathcal{G}$ and $\pi_2^*\mathcal{F} \to \pi_2^*\mathcal{G}$ commute with the isomorphisms ψ. There is no extra cocycle condition.

In general, there is a functor from quasicoherent $\mathcal{O}_X$-modules to descent data taking $\mathcal{E}$ to $f^*\mathcal{E}$, and defining ψ in the obvious manner.

Theorem 3 (Faithfully flat descent)**.** *Let $f : Y \to X$ be a faithfully flat, quasicompact morphism. Then the natural functor from quasicoherent $\mathcal{O}_X$-modules to descent data for quasicoherent sheaves defined by f is an equivalence of categories.*

The reference for this is SGA 1, Exposé VIII, section 1. However, the proof there is written in a somewhat cryptic manner; we will see a somewhat simplified proof in the exercises.

Note that faithfully flat descent for quasicoherent sheaves includes as a special case *Galois descent*: if L/K is a finite Galois extension of fields, and V is an L-vector space equipped with a semilinear action of $\mathrm{Gal}(L/K)$, then V has a basis of invariant elements. (The usual proof uses Noether's nonabelian generalization of Hilbert's Theorem 90, i.e., the fact that the first Galois cohomology set of $\mathrm{Gal}(L/K)$ acting on $\mathrm{GL}_n(L)$ is trivial.)

Armed with faithfully flat descent for quasicoherent sheaves, one can now establish descent for various properties of morphisms. (Some of these can be found in EGA 4, part 2.) For example:

Theorem 4. *Let $f : Y \to X$ be a morphism, and let $g : Z \to X$ be a faithfully flat quasicompact morphism. Then f is of finite type if and only if the base change of f by g is of finite type.*

Proof. Suppose first that $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, $Z = \mathrm{Spec}(C)$ are all affine, and that the base change of f by g is of finite type. Then B is the direct limit of its finitely generated A-subalgebras B_i, and so $B \otimes_A C$ is the direct limit of the finitely generated C-subalgebras

$B_i \otimes_A C$. By hypothesis, $B \otimes_A C$ is finitely generated as a C-algebra; each generator can itself be written in terms of finitely many elements of B and C. Hence $B \otimes_A C$ can be generated over C by finitely many elements of B, and so must occur as one of the $B_i \otimes_A C$. For that index i, the fact that the inclusion $B_i \to B$ is an isomorphism follows from the fact that $B_i \otimes_A C \to B \otimes_A C$ is an isomorphism because C was assumed to be faithfully flat over A.

To finish, we must show that if the base change of f by g is quasicompact, then f is quasicompact. We may assume X is affine, as then is Z because g was required to be quasicompact, as then is $Y \times_X Z$ by hypothesis. Let $\{U_i\}$ be an open affine cover of Y. By hypothesis, the open cover $\{U_i \times_X Z\}$ of $Y \times_X Z$ admits a finite subcover. Since those $Z \to X$ is surjective, the corresponding U_i must then cover Y. Hence Y is a union of finitely many affines, hence quasicompact. $\square$