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18.726 Algebraic Geometry  
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18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)  
Divisors on curves and Riemann-Roch (updated 31 Mar 09)

We continue the discussion of divisors but now restricted to curves. Again, see IV.1 for Riemann-Roch and IV.2 for Riemann-Hurwitz.

## 1 The Riemann-Roch theorem

Again, let  $X$  be a (projective, irreducible, nonsingular) curve over an algebraically closed field  $k$ . Since  $X$  is one-dimensional, the canonical sheaf  $\omega_{X/k}$  coincides with the sheaf of Kähler differentials  $\Omega_{X/k}$ . By a *canonical divisor*, I mean a divisor  $K$  defined by any meromorphic section of  $\omega_{X/k}$ . (This means that a canonical divisor is in fact not canonical in any sense. Sorry about that.)

As in the elliptic curve example, there is a homomorphism  $\text{Div } X \rightarrow \mathbb{Z}$  sending  $(P)$  to 1 for each  $P \in X(k)$ , and this factors through  $\text{Cl } X$  because any principal divisor has degree 0 (Hartshorne, Corollary II.6.10).

Write  $l(D)$  as shorthand for  $\dim_k \Gamma(X, \mathcal{L}(D))$ . The following theorem will be proved later using properties of sheaf cohomology (particularly Serre duality), but in the meantime we will see (in this lecture and in the next problem set) how it tells us many useful things that have no overt relationship to cohomology.

**Theorem** (Riemann-Roch). *There exists a nonnegative integer  $g = g(X)$  with the following property. For any divisor  $D$  and any canonical divisor  $K$ ,*

$$l(D) - l(K - D) = \deg(D) + 1 - g.$$

**Corollary.** *The integer  $g$  in Riemann-Roch can be identified as*

$$g = l(K) = \dim_k \Gamma(X, \Omega_{X/k}).$$

*Proof.* Take  $D = 0$ . Then  $l(D) = 1$  because any global regular function on a curve (or indeed on any projective variety) is constant. This forces  $l(K) = g$ .  $\square$

The quantity  $l(K)$  is called the *genus* of  $K$ , or more precisely the *geometric genus*. In case  $k = \mathbb{C}$ , this will end up matching the topological genus of the Riemann surface associated to  $X$ .

**Corollary.** *The integer  $g$  in Riemann-Roch can also be identified by the formula*

$$\deg(K) = 2g - 2.$$

*Proof.* Apply Riemann-Roch with  $D = K$  to obtain (by the previous corollary)

$$g - 1 = l(K) - l(0) = \deg(K) + 1 - g.$$

$\square$

**Corollary.** *If  $\deg(D) > 2g - 2$ , or  $\deg(D) = 2g - 2$  and  $D \not\sim K$ , then*

$$l(D) = \deg(D) + 1 - g \geq g - 1.$$

*Proof.* If  $\deg(D) = 2g - 2$ , then  $\deg(K - D) = 0$ . If  $f \in K(X)$  nonzero satisfies  $(f) + K - D \geq 0$ , we must have equality because the left side has degree 0. Thus  $l(K - D)$  is only nonzero if  $K \sim D$ .

If  $\deg(D) > 2g - 2$ , then  $\deg(K - D) < 0$ . In this case,  $(f) + K - D$  has negative degree and so cannot be effective, so  $l(K - D) = 0$  no matter what.  $\square$

**Corollary.** *For  $g \geq 2$ , for any divisor  $D$  of degree at least  $2g - 1$ , the complete linear system associated to  $D$  defines a closed immersion of  $D$  into a projective space.*

## 2 The canonical (almost) embedding

The *canonical embedding* is the map to projective space defined by the complete linear system associated to a canonical divisor  $K$ . The name suggests that it is always a closed immersion, but this is only *almost* true; there are a few exceptions in low genus (for which see the exercises).

**Lemma.** *For any point  $P$  and any divisor  $D$ , we have*

$$l(D) \leq l(D + P) \leq l(D) + 1.$$

*Consequently,  $l(D) \leq \deg(D) + 1$ .*

*Proof.* We have an exact sequence of sheaves

$$0 \rightarrow \mathcal{L}(D) \rightarrow \mathcal{L}(D + (P)) \rightarrow \mathcal{E} \rightarrow 0$$

where  $\mathcal{E}$  is the quotient of  $\mathcal{O}_X$  by the ideal sheaf defining  $P$ . So clearly  $l(D) \leq l(D + P)$ . On the other hand, taking global sections yields a short exact sequence

$$0 \rightarrow \Gamma(X, \mathcal{L}(D)) \rightarrow \Gamma(X, \mathcal{L}(D + (P))) \rightarrow \Gamma(X, \mathcal{E})$$

and the last term is one-dimensional over  $k$ , so we get  $l(D + P) \leq l(D) + 1$ .  $\square$

**Proposition.** *The canonical embedding is a closed immersion if and only if  $X$  is not hyperelliptic.*

*Proof.* The special cases  $g = 2, 3$  are discussed in the problem set, so I'll only sketch the general argument. Put  $D = (P) + (Q)$  for  $P, Q \in X(k)$  not necessarily distinct. We need to check whether we always have

$$l(K - D) = l(K) - 2 = g - 2.$$

By Riemann-Roch,

$$l(K - D) = l(D) + g - 3$$

so we have an embedding if and only if  $l(D) = 0$  for any effective  $D$  of degree 2; but a failure of that defines a two-to-one map to  $\mathbb{P}^1$ , in which case  $X$  is hyperelliptic. (Strictly speaking, we should also check for  $D$  of degree 1, but it's easy to see that if such  $D$  has  $l(D) > 0$ , then there exists a rational function on  $X$  with a single pole, which gives a degree 1 map to  $\mathbb{P}^1$ . That is,  $X \cong \mathbb{P}^1$ .)  $\square$

The canonical embedding, and variants of it (e.g., using higher multiples of a canonical divisor) are key tools for studying the *moduli space of curves* of a given genus. This is “almost” a scheme  $M_g$  which represents the functor taking schemes to families of curves of genus  $g$ , except that this functor is not quite representable. It becomes representable in the category of *Deligne-Mumford stacks*, which extend schemes in much the same way that *orbifolds* extend manifolds (by allowing quotients by finite group actions).

### 3 The Riemann-Hurwitz formula

Let  $f : X \rightarrow Y$  be a finite *separable* morphism of curves (i.e., the induced field extension  $k(X)/k(Y)$  is separable). The *ramification divisor* of  $f$  is defined as

$$R = \sum_{P \in X(k)} \text{length}(\Omega_{X/Y})_P(P),$$

where as usual  $\Omega_{X/Y}$  is the module of Kähler differentials.

**Proposition.** *We have*

$$K_X \sim f^*K_Y + R.$$

*Proof.* (Compare Hartshorne Proposition IV.2.3.) Note that

$$0 \rightarrow f^*\Omega_{Y/k} \rightarrow \Omega_{X/k} \rightarrow \Omega_{X/Y} \rightarrow 0$$

is exact; this follows from properties of Kähler differentials except for the injectivity on the left. But that we can check at generic points, where it follows because  $k(X)$  is separable over  $k(Y)$ .

We can then tensor with  $\Omega_{X/k}^\vee$  to obtain another exact sequence

$$0 \rightarrow (f^*\Omega_{Y/k}) \otimes \Omega_{X/k}^\vee \rightarrow \mathcal{O}_X \rightarrow \Omega_{X/Y} \otimes \Omega_{X/k}^\vee \rightarrow 0.$$

However,  $\Omega_{X/Y}$  is supported on finitely many points, so it is isomorphic to its twist by  $\Omega_{X/k}^\vee$ . So we really have an isomorphism

$$(f^*\Omega_{Y/k}) \otimes \Omega_{X/k}^\vee \cong \mathcal{O}_X/\Omega_{X/Y}.$$

We thus get an equality of associated divisors; these are  $f^*K_Y - K_X$  on the left and  $-R$  on the right.  $\square$

Using Riemann-Roch, we deduce the *Riemann-Hurwitz formula*.

**Proposition.** *We have*

$$2g(X) - 2 = (\deg(f))(2g(Y) - 2) + \deg(R),$$

where  $\deg(f)$  is the degree of  $f$  (i.e., the degree of the field extension  $k(X)/k(Y)$ ).

Moreover, the contribution of  $P \in X(k)$  can sometimes be computed very simply. Namely, put  $Q = f(P)$ , and pick  $t \in k(Y)$  which generates  $\mathfrak{m}_{Y,Q}$ ; then  $f^*(t)$  generates  $\mathfrak{m}_{X,P}^e$  for some nonnegative integer  $e$ . We call  $e = e_P$  the *ramification index* of  $P$ . Then

$$\text{length}(\Omega_{X/Y})_P \geq e_P - 1,$$

with equality if and only if  $f$  is *tamely ramified*, i.e.,  $e_P$  is not divisible by the characteristic of  $k$ .

In case  $k = \mathbb{C}$ , the Riemann-Hurwitz formula has a topological meaning: the quantity  $2 - 2g(X)$  turns out to compute the *Euler characteristic* of the associated Riemann surface. The Euler characteristic (computed using homology, or compactly supported cohomology) is an *additive* invariant of a topological space. If the map  $f$  were unramified, then we would have  $\deg(R) = 0$  and the space  $X$  would have Euler characteristic equal to  $\deg(f)$  times that of  $Y$ . Otherwise, one must subtract  $e_P - 1$  for each point  $P$  with  $e_P > 1$ , because you get  $X$  from an unramified cover of  $Y$  by removing  $e_P$  different points from the fibre (each of which has Euler characteristic 1) and adding one point back in.