18.726 Algebraic Geometry
Spring 2009

18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Cohomology of quasicoherent sheaves (updated 25 Apr 09)

There is one more fundamental theorem about affine schemes! Here it is.

1 The theorem, and a bogus proof

Let's start with the statement of the *fourth fundamental theorem of affine schemes.*

Theorem. *Let X be an affine scheme and let $\mathcal{F}$ be a quasicoherent sheaf on X. Then $H^i(X, \mathcal{F}) = 0$ for $i > 0$, that is, $\mathcal{F}$ is acyclic for sheaf cohomology.*

Here is how *not* to prove this theorem.

Bogus proof of the theorem. Put $X = \operatorname{Spec} A$. From the earlier fundamental theorems of affine schemes, we know we can write $\mathcal{F} = \tilde{M}$ for some A-module M. Since $\underline{\mathrm{Mod}}_A$ has enough injectives, we can find a monomorphism $M \to I$ with I an injective A-module. Put $N = I/M$. Again by the earlier fundamental theorems of affine schemes, we know that the exact sequence

$$0 \to M \to I \to I/M \to 0$$

of A-modules is precisely what you get by taking global sections of the exact sequence

$$0 \to \tilde{M} \to \tilde{I} \to I/\tilde{M} \to 0.$$

So in the long exact sequence in cohomology, the connecting homomorphism into $H^1(X, \tilde{M})$ is zero. On the other hand, $H^i(X, \tilde{I}) = 0$ for all $i > 0$ since $\tilde{I}$ is injective, so $H^1(X, \tilde{M})$ is forced to be zero. Moreover, for $i > 1$, $H^i(X, \tilde{M}) \cong H^{i-1}(X, \widetilde{I/M})$, so we may prove the higher vanishing by dimension shifting. □

What's wrong with this proof? The problem is that while the injectivity of I in $\underline{\mathrm{Mod}}_A$ implies the injectivity of I in the category of *quasicoherent* $\mathcal{O}_X$-modules, it does not imply injectivity in the category of arbitrary $\mathcal{O}_X$-modules, or of sheaves of abelian groups on X. In particular, it is unclear whether injectivity of I implies that I is flasque. One can at least show that I is "basically flasque", in that the restriction from $\Gamma(X, I) = I$ to $\Gamma(D(f), I) = I_f$ is surjective, but this isn't really enough to do anything useful.

There are two ways to fix this. One way (used in Hartshorne, and also in the book by Ueno that I recommended earlier) is to restrict attention to noetherian rings, and then prove that an injective module does indeed give rise to a flasque sheaf. The other way (used in EGA) is to compute with Čech cohomology instead of sheaf cohomology, so that you can deal only with finite covers by distinguished opens. That's what I'll do here.

First, however, I should mention that there is an easy argument to show that H^1 vanishes. The following is close to the *third* fundamental theorem of affine schemes; see Hartshorne Proposition II.5.6 for the proof. (Since I won't use this to prove the theorem, you may instead view it as a corollary of the theorem.)

Lemma. *Let $X = \operatorname{Spec} A$ be an affine scheme. Let*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

be an exact sequence of $\mathcal{O}_X$-modules such that $\mathcal{F}_1$ is quasicoherent (but don't assume anything about the other two). Then the sequence

$$0 \to \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}_2) \to 0$$

is exact.

This implies that the connecting homomorphism $H^0(X, \mathcal{F}_2) \to H^1(X, \mathcal{F}_1)$ is zero, so $H^1(X, \mathcal{F}_1)$ injects into $H^1(X, \mathcal{F})$. If we then choose $\mathcal{F}$ to be injective, we deduce $H^1(X, \mathcal{F}_1) = 0$.

2 Applications

Before proving the theorem, let me mention some corollaries. First, from the Čech cohomology discussion, we deduce the following.

Corollary. *Let X be a scheme and let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of X. Suppose that for each finite subset $J \subseteq I$, the intersection $U_J = \cap_{j \in J} U_j$ is affine. Then for any quasicoherent sheaf $\mathcal{F}$ on X, the sheaf cohomology of $\mathcal{F}$ is computed by the Čech cohomology for the cover $\mathfrak{U}$; that is,*

$$H^i(X, \mathcal{F}) = \check{H}^i(\mathfrak{U}, \mathcal{F}).$$

Recall that inside a *separated* scheme, the intersection of any two open affines is again affine. We thus have the following; I'll illustrate next time by using this to compute the cohomology of the sheaves $\mathcal{O}(n)$ on projective space.

Corollary. *Let X be a separated scheme and let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open affine cover of X. Then for any quasicoherent sheaf $\mathcal{F}$ on X,*

$$H^i(X, \mathcal{F}) = \check{H}^i(\mathfrak{U}, \mathcal{F}).$$

Here is an even more specialized corollary, which in itself is not so useful. I mention it because I will prove this directly and use it as a lemma to prove the whole theorem.

Corollary. *Let A be a ring and choose $f_1, \dots, f_n \in A$ which generate the unit ideal. Let $\mathfrak{U}$ be the open cover of $X = \operatorname{Spec} A$ by $D(f_i)$ for $i = 1, \dots, n$. Then for any A-module M, $\check{H}^0(\mathfrak{U}, \tilde{M}) = M$ and $\check{H}^i(\mathfrak{U}, \tilde{M}) = 0$ for $i > 0$.*

3 A correct proof of the theorem

Following Grothendieck (and I think Serre before him, in the context of varieties), we will prove the last corollary first, and then use that to prove the theorem. So our first order of business is to show that the complex

$$0 \to M \to \check{C}^0(\mathfrak{U}, \tilde{M}) \to \check{C}^1(\mathfrak{U}, \tilde{M}) \to \cdots$$

is exact. Remember that this complex came from the complex of sheaves

$$0 \to \tilde{M} \to \check{\mathcal{C}}^0(\mathfrak{U}, \tilde{M}) \to \check{\mathcal{C}}^1(\mathfrak{U}, \tilde{M}) \to \cdots$$

by taking global sections. We proved in the Čech cohomology lecture that this sequence of sheaves is exact (by computing at stalks). Moreover, each of the constituent sheaves is quasicoherent, for the following reason. Each sheaf in the sequence equals a direct sum of sheaves each of the form $j_{U*}(\tilde{M}|_U)$ for U an intersection of some of the U_j. In particular, each such intersection has the form $D(g)$ for some $g \in A$. But this sheaf is simply the quasicoherent sheaf associated to the A-module M_g.

Since we have an exact sequence of quasicoherent sheaves, taking global sections gives us an exact sequence of A-modules. This proves the corollary. So now we know that for any finite cover $\mathfrak{U}$ of $\operatorname{Spec} A$ by distinguished opens,

$$\check{H}^0(\mathfrak{U}, \tilde{M}) = M, \qquad \check{H}^i(\mathfrak{U}, \tilde{M}) = 0 \quad (i > 0).$$

The same holds if we take the direct limit over finite covers by distinguished opens. However, this gives the same result as taking the direct limit over *all* open covers because any cover can be refined to a finite cover by distinguished opens. We conclude that

$$\check{H}^0(X, \tilde{M}) = M, \qquad \check{H}^i(X, \tilde{M}) = 0 \quad (i > 0);$$

although the theorem that says that the direct limit Čech cohomology also computes sheaf cohomology doesn't apply (because X is not Hausdorff), one can still show that this implies

$$H^0(X, \tilde{M}) = M, \qquad H^i(X, \tilde{M}) = 0 \quad (i > 0)$$

using the following theorem of Cartan, applied with B being the collection of distinguished open affines.

Theorem (Cartan)**.** *Let X be a topological space. Let B be a basis of X closed under pairwise intersections. Let $\mathcal{F}$ be a sheaf of abelian groups on X such that $\check{H}^i(U, \mathcal{F}) = 0$ for all $U \in B$. Then $\check{H}^i(X, \mathcal{F})$ is naturally isomorphic to $H^i(X, \mathcal{F})$ for all $i \geq 0$.*

We will prove this in the next section. It can also be proved using spectral sequences; see the optional handout.

4 Comparison of Čech and sheaf cohomology

Before proving Cartan's theorem, here is a lemma which generalizes a fact we already know about flasque sheaves.

Lemma. *Let X be a topological space. Let $\mathcal{F}$ be a sheaf of abelian groups on X such that $\check{H}^1(X, \mathcal{F}) = 0$. Then for any short exact sequence*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of sheaves,

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H}) \to 0$$

is exact.

Proof. (proof suggested by Fucheng Tan) We need only check surjectivity on the right. Let $s \in \Gamma(X, \mathcal{H})$ be any section; let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of X such that for each $i \in I$, $s|_{U_i}$ lifts to a section $t_i \in \Gamma(U_i, \mathcal{G})$. For $i, j \in I$, put

$$u_{ij} = t_i|_{U_i \cap U_j} - t_j|_{U_i \cap U_j} \in \Gamma(U_i \cap U_j, \mathcal{G}).$$

Since u_{ij} has zero image in $\Gamma(U_i \cap U_j, \mathcal{G})$, we may also view as an element of $\Gamma(U_i \cap U_j, \mathcal{F})$. With this convention, we see that the u_{ij} form a Čech 1-cocycle of $\mathcal{F}$ for the open cover $\mathfrak{U}$.

Before proceeding, note that there is a natural way to replace the above data for one cover $\mathfrak{U}$ with a refinement $\mathfrak{V} = \{V_j\}_{j \in J}$. Namely, the refinement comes by definition with a map $\lambda : J \to I$ such that $V_j \subseteq U_{\lambda(j)}$ for each j. To pass from $\mathfrak{U}$ to $\mathfrak{V}$:

- replace the collection of the t_i for $i \in I$ with the collection of the $t_{\lambda(j)}|_{V_j}$ for $j \in I$;
- replace the collection of the t_{ij} for $i, j \in I$ with the collection of the $u_{\lambda(k)\lambda(l)}|_{V_k \cap V_j}$ for $k, l \in I$.

To avoid excess notation, we will speak of "replacing $\mathfrak{U}$ by a refinement" which will also be labeled $\mathfrak{U}$.

Since $\check{H}^1(X, \mathcal{F}) = 0$ by hypothesis, we can replace $\mathfrak{U}$ by a refinement in such a way that u_{ij} become a Čech 1-coboundary. This means that there are sections $v_i \in \Gamma(U_i, \mathcal{F})$ such that

$$v_i|_{U_i \cap U_j} - v_j|_{U_i \cap U_j} = u_{ij} \qquad (i, j \in I).$$

For $i \in I$, we now form

$$w_i = t_i - v_i \in \Gamma(U_i, \mathcal{G}).$$

These sections have the property that on one hand, the image of w_i in $\Gamma(U_i, \mathcal{H})$ equals $s|_{U_i}$ (since v_i, having come from $\mathcal{F}$, maps to zero in $\mathcal{H}$), and on the other hand,

$$\begin{aligned} w_i|_{U_i \cap U_j} - w_j|_{U_i \cap U_j} &= (t_i|_{U_i \cap U_j} - v_i|_{U_i \cap U_j}) - (t_j|_{U_i \cap U_j} - v_j|_{U_i \cap U_j}) \\ &= (t_i|_{U_i \cap U_j} - t_j|_{U_i \cap U_j}) - (v_i|_{U_i \cap U_j} - v_j|_{U_i \cap U_j}) \\ &= u_{ij} - u_{ij} = 0. \end{aligned}$$

Hence the w_i glue to a section $w \in \Gamma(X, \mathcal{G})$ lifting s, as desired. □

Proof of Cartan's theorem. The claim is true for $i = 0$ because of the sheaf axiom. We use this as a basis for induction on i, using dimension shifting. Assume that for some $i > 0$, the claim is true for every value less than i. Choose a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

with $\mathcal{G}$ flasque. By the previous lemma, for any $U \in B$,

$$0 \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G}) \to \Gamma(U, \mathcal{H}) \to 0$$

is exact. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of X by basic open sets. Since B is closed under pairwise intersections, it follows that

$$0 \to \check{C}^{\cdot}(\mathfrak{U}, \mathcal{F}) \to \check{C}^{\cdot}(\mathfrak{U}, \mathcal{G}) \to \check{C}^{\cdot}(\mathfrak{U}, \mathcal{H}) \to 0$$

is an exact sequence of complexes. Since every open cover can be refined to an open cover by basic opens, taking direct limits over all covers is the same as taking direct limits over basic open covers, which means that

$$0 \to \check{C}^{\cdot}(X, \mathcal{F}) \to \check{C}^{\cdot}(X, \mathcal{G}) \to \check{C}^{\cdot}(X, \mathcal{H}) \to 0$$

is again an exact sequence of complexes. The same holds if we replace X by any basic open set U, by the same reasoning.

We now take the long exact sequence in cohomology associated to this short exact sequence of complexes, and compare it to the long exact sequence in sheaf cohomology. We get the diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \check{H}^{i-1}(X, \mathcal{G}) & \longrightarrow & \check{H}^{i-1}(X, \mathcal{H}) & \longrightarrow & \check{H}^{i}(X, \mathcal{F}) & \longrightarrow & \check{H}^{i}(X, \mathcal{G}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^{i-1}(X, \mathcal{G}) & \longrightarrow & H^{i-1}(X, \mathcal{H}) & \longrightarrow & H^{i}(X, \mathcal{F}) & \longrightarrow & H^{i}(X, \mathcal{G}) & \longrightarrow & \cdots
\end{array}$$

We first notice that $\check{H}^i(X, \mathcal{G}) = H^i(X, \mathcal{G}) = 0$ because $\mathcal{G}$ is flasque. If we replace X by a basic open set U and then look at the top sequence, we see that for $i > 1$, $\check{H}^{i-1}(U, \mathcal{H})$ is sandwiched between two zero groups, so it is also zero. That is, $\mathcal{H}$ also satisfies the hypothesis of the theorem.

We may now argue by dimension shifting as follows. The first vertical arrow is an isomorphism (for $i = 1$ this holds by the sheaf axiom, otherwise both groups vanish), the second vertical arrows is an isomorphism by the induction hypothesis, and the fourth vertical arrow is the zero map between zero groups. The five lemma thus implies that the third arrow is an isomorphism. $\square$