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18.726 Algebraic Geometry
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18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Hilbert polynomials and flatness (revised 17 Apr 09)

See Hartshorne III.9 again.

1 Hilbert polynomials

Let k be a field (not necessarily algebraically closed). Let $j : X \rightarrow \mathbb{P}_k^r$ be a closed immersion for some $r \geq 1$. Write $\mathcal{O}_X(1)$ for the inverse image by j of the twisting sheaf $\mathcal{O}(1)$. Let \mathcal{F} be a finitely generated quasicoherent sheaf on X .

The *Euler characteristic* of \mathcal{F} is defined as

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathcal{F});$$

we know from Serre's finiteness theorem that each summand is finite, and we also know that there are no terms in dimension greater than r . So this is indeed a well-defined integer.

Lemma. *The Euler characteristic is additive in short exact sequences; that is, if*

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$$

is exact, then

$$\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H}).$$

Proof. Exercise in the long exact sequence in cohomology. □

Corollary. *If*

$$0 \rightarrow \mathcal{F}_1 \rightarrow \cdots \rightarrow \mathcal{F}_n \rightarrow 0$$

is an exact sequence of finitely generated quasicoherent sheaves, then

$$\sum_{i=1}^n (-1)^i \chi(X, \mathcal{F}_i) = 0.$$

Theorem. *There exists a polynomial $P(z) \in \mathbb{Q}[z]$ such that*

$$\chi(X, \mathcal{F}(n)) = P(n) \quad (n \in \mathbb{Z}).$$

Moreover, the degree of P is at most the dimension of X .

Proof. By replacing \mathcal{F} by $j_*\mathcal{F}$, we may reduce to the case $X = \mathbb{P}_k^r$. Also, changing the base field doesn't change any of the dimensions (e.g., by looking at Čech cohomology; this is a special case of the *flat base change theorem*), so we may assume k is algebraically closed.

We proceed by induction on the dimension of the support of \mathcal{F} . If that support is empty (i.e., \mathcal{F} is the zero sheaf), then obviously $P(z) = 0$ works.

Otherwise, form an exact sequence

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{F}(-1) \xrightarrow{\times x_r} \mathcal{F} \rightarrow \mathcal{H} \rightarrow 0$$

and note that \mathcal{G} and \mathcal{H} have support of lower dimension than \mathcal{F} provided that we ensure that the hyperplane $x_r = 0$ does not contain any component of the support of \mathcal{F} . (We can arrange this given that k is algebraically closed; see exercises. In fact, k infinite would be sufficient.) By the induction hypothesis, we know that $\chi(\mathbb{P}_k^r, \mathcal{F}(n)) - \chi(\mathbb{P}_k^r, \mathcal{F}(n-1))$ is a polynomial in n of degree at most $\dim(\mathbf{Supp} \mathcal{F}) - 1$. It is an elementary exercise in algebra to then see that $\chi(\mathbb{P}_k^r, \mathcal{F}(n))$ is a polynomial in n of degree at most $\dim(\mathbf{Supp} \mathcal{F})$. \square

The polynomial $P(n)$ is called the *Hilbert polynomial* of the sheaf \mathcal{F} ; in case $\mathcal{F} = \mathcal{O}_X$, we call it the *Hilbert polynomial* of the scheme X itself. Note that by Serre's vanishing theorem, for some n_0 , we have

$$P(n) = \dim_k H^0(X, \mathcal{F}) \quad (n \geq n_0);$$

this was the original definition of the Hilbert polynomial.

For example, the Hilbert polynomial of \mathbb{P}_k^r itself is $P(n) = \binom{n+r}{n}$. For another example, the Hilbert polynomial of the subscheme $\mathrm{Spec} k[x]/(x^2)$ of \mathbb{P}_k^1 is $P(n) = 2$, which is the same as the Hilbert polynomial of a scheme consisting of two distinct reduced points. This is suggestive, because this scheme can indeed be written as a flat limit of pairs of distinct points.

2 Flatness and Hilbert polynomials

The Hilbert polynomial can be used to give the following numerical criterion for flatness. (The locally noetherian hypothesis is important; I think one can replace “integral” by “connected and reduced”.)

Theorem. *Let T be an integral (locally) noetherian scheme. Let $X \subseteq \mathbb{P}_T^r$ be a closed subscheme. Let \mathcal{F} be a coherent sheaf on X . For each $t \in T$, let $P_t \in \mathbb{Q}[z]$ be the Hilbert polynomial of the pullback of \mathcal{F} to the fibre X_t viewed as a subscheme of $\mathbb{P}_{\kappa(t)}^r$ (where $\kappa(t) = \mathcal{O}_{T,t}/\mathfrak{m}_{T,t}$ is the residue field of the point t). Then \mathcal{F} is flat relative to $X \rightarrow T$ if and only if P_t is constant as a function of t .*

In particular, X itself is flat over T if and only if the Hilbert polynomial of X_t is constant as t varies. This gives us a way to check whether a morphism is flat which we were sorely lacking before.

Proof. (Compare Hartshorne Theorem III.9.9, or EGA III §7.9.) We first note that we can reduce to the case $X = \mathbb{P}_T^r$ by replacing \mathcal{F} with its direct image. We next note that it suffices to consider the case where $T = \mathrm{Spec} A$ for A a local integral noetherian ring.

We then show that \mathcal{F} is flat over T if and only if $H^0(X, \mathcal{F}(m))$ is finite free over A for m sufficiently large. On one hand, if \mathcal{F} is flat over T , then so are all the terms in the sheafy

Čech resolution of $\mathcal{F}(m)$ for the usual open cover \mathfrak{U} (since open immersions are flat). Taking global sections, we see that the terms of the exact sequence

$$0 \rightarrow H^0(X, \mathcal{F}(m)) \rightarrow \check{C}^0(\mathfrak{U}, \mathcal{F}(m)) \rightarrow \cdots \rightarrow \check{C}^r(\mathfrak{U}, \mathcal{F}(m)) \rightarrow 0$$

are all flat except possibly for the first term. This then implies flatness of $H^0(X, \mathcal{F}(m))$ (exercise). Since it's also finitely generated over A by Serre's finiteness theorem, it is free.

On the other hand, if we pick m_0 such that $H^0(X, \mathcal{F}(m))$ is finite free over A for $m \geq m_0$, then we can recover \mathcal{F} as \tilde{M} for

$$M = \bigoplus_{m \geq m_0} H^0(X, \mathcal{F}(m)).$$

Since M is flat, so is \mathcal{F} .

We now need to show that $H^0(X, \mathcal{F}(m))$ is finite free for m large if and only if P_t is constant. I claim that this follows by checking

$$H^0(X_t, \mathcal{F}_t(m)) = H^0(X, \mathcal{F}(m)) \otimes_A \kappa(t)$$

for m large (even if I don't prove this uniformly in t). Namely, if $H^0(X, \mathcal{F}(m))$ is finite free over A for $m \geq m_0$, then for each t , for m large, I find that P_t equals P_η for η the generic point of T . On the other hand, if P_t is the same for the generic point and the closed point, then I can make m large enough to work for both, and obtain finite freeness of $H^0(X, \mathcal{F}(m))$.

To check

$$H^0(X_t, \mathcal{F}_t(m)) = H^0(X, \mathcal{F}(m)) \otimes_A \kappa(t),$$

we may reduce to the case where t is the closed point by replacing A with $\mathcal{O}_{T,t}$. Since A is noetherian, we can find a short exact sequence

$$A^{\oplus n} \rightarrow A \rightarrow \kappa(t) \rightarrow 0$$

of A -modules. We can then tensor with \mathcal{F} to get an exact sequence; it follows (exercise) that

$$H^0(X, \mathcal{F}(m)^{\oplus n}) \rightarrow H^0(X, \mathcal{F}(m)) \rightarrow H^0(X, \mathcal{F}_t(m)) \rightarrow 0$$

is exact for m large. I can pull out the direct sum, and then we read off what we want. \square

3 Hilbert schemes

It turns out that there is a universal family of closed subschemes of projective space with a fixed Hilbert polynomial.

Theorem (Grothendieck). *Fix a field k and an integer r . Let $P(z) \in \mathbb{Q}[z]$ be a polynomial. There exists a noetherian scheme H over $\text{Spec } k$ and a closed subscheme X of \mathbb{P}_H^r which is flat with Hilbert polynomial $P(z)$, such that for any noetherian scheme T and any closed subscheme Y of \mathbb{P}_T^r which is flat with Hilbert polynomial $P(z)$, there is a unique morphism $T \rightarrow H$ such that $Y = X \times_H T$ as closed subschemes of $\mathbb{P}_T^r \cong \mathbb{P}_H^r \times_H T$.*

For instance, one can show that a closed subscheme of \mathbb{P}_k^r is a d -dimensional plane if and only if it has Hilbert polynomial $P(n) = \binom{n+d-1}{n}$. The parameter scheme in this case is the Grassmannian of d -dimensional planes in \mathbb{P}_k^r .

4 Hilbert polynomials, degree, and dimension

Some of the basic information contained in the Hilbert polynomial of a scheme is the following.

Lemma. *Let $P(z)$ be the Hilbert polynomial of a closed subscheme X of \mathbb{P}_k^n .*

- (a) *We have $\deg(P) = \dim(X)$.*
- (b) *Put $d = \dim(X)$. For any d -dimensional plane $P \subseteq \mathbb{P}_k^n$ such that $\dim(X \cap P) = 0$, the length of $X \cap P$ is $d!$ times the leading coefficient of P . (This length is called the degree of X .)*

Proof. We may assume k is algebraically closed. We first need to know that for a *generic* d -dimensional plane P (i.e., one chosen outside some closed subscheme of the Grassmannian), we have $\dim(X \cap P) = 0$. This follows from the fact that as long as $X \neq \emptyset$, for a generic hyperplane H , we have $\dim(X \cap H) < \dim(X)$ (exercise).

Put $\mathcal{F} = j_*\mathcal{O}_X$ for $j : X \rightarrow \mathbb{P}_k^n$ the given closed immersion. For H a hyperplane with $\dim(X \cap H) < \dim(X)$, we have an exact sequence

$$0 \rightarrow \mathcal{F}(-1) \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$$

where \mathcal{G} is the direct image of the structure sheaf of $X \cap H$. If $P(z)$ is the Hilbert polynomial of X , it follows that the Hilbert polynomial of $X \cap H$ is $P(z) - P(z - 1)$. From this, both claims follow. \square