18.726 Algebraic Geometry
Spring 2009

18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
GAGA (updated 30 Apr 2009)

We now discuss a classic theorem of algebraic geometry, Serre's GAGA, which exposes a tight relationship between algebraic geometry over the complex numbers and complex analytic geometry. By far the best reference for this is Serre's original paper *Géométrie algébrique et géométrie analytique*. (Thanks to Bjorn Poonen for reporting some errors, which have now been corrected.)

1 Coherent sheaves

In order to discuss GAGA, I need to talk about coherent sheaves not just on schemes, but on analytic spaces. In fact, the notion is well-defined on any locally ringed space.

Let $(X, \mathcal{O}_X)$ be a locally ringed space. We say a sheaf $\mathcal{F}$ is *coherent* if $\mathcal{F}$ is finitely generated, and for any open subset U of X, any nonnegative integer n, and any homomorphism $h : \mathcal{O}_X^{\oplus n}|_U \to \mathcal{F}|_U$, the kernel of h is itself finitely generated. **Warning!** I originally only required this for h surjective, but I don't think that is enough. (Important note: we don't require the kernel to be generated by finitely many sections over all of U.) This is stronger than saying that $\mathcal{F}$ is finitely presented, in which case we only require that *one* such surjection h must have this property. In particular, $\mathcal{O}_X$ itself need not be coherent.

However, if X is locally noetherian, then all finitely generated quasicoherent sheaves are in fact coherent. This follows from the following result.

Theorem. *Let A be a* noetherian *ring, put $X = \operatorname{Spec} A$, let V be an open subset of X, and let $\mathcal{F}$ be an $\mathcal{O}_X|_V$-module. Then the following are equivalent.*

(a) $\mathcal{F}$ is coherent.

(b) $\mathcal{F}$ is finitely generated and quasicoherent.

(c) We have $\mathcal{F} = \tilde{M}$ for some finitely generated A-module M.

I'm only going to show this for $V = X$, as this is the only case I need. For the general case, see EGA 1, Théorème 1.5.1.

Proof. Even without a noetherian hypothesis, it is obvious that (a) implies (b), and we checked (b) implies (c) in a previous lecture.

To check that (c) implies (a) under the noetherian hypothesis, note that the claim is local, so it suffices to check that the kernel of a homomorphism $\mathcal{O}_X^{\oplus n}|_{D(f)} \to \mathcal{F}|_{D(f)}$ is finitely generated. It is represented by a homomorphism $A_f^n \to M_f$ of A_f-modules, but A_f is noetherian since A is. Hence the kernel of the homomorphism, being a submodule of a finitely generated A_f-module, is itself a finitely generated A_f-module (because A is noetherian). $\square$

Lemma. *Let*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

be a short exact sequence of quasicoherent sheaves on a locally ringed space X. Then if any two of $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ are coherent, so is the third.

Proof. Exercise. □

Beware that it is not obvious that the inverse image of a coherent sheaf is coherent, since the defining condition involves looking at all open subsets.

2 Analytification of coherent sheaves

In order to state the GAGA theorems, we use the fact that there is a morphism of locally ringed spaces

$$h : \tilde{\mathbb{P}}^r_{\mathbb{C}} \to \mathbb{P}^r_{\mathbb{C}},$$

where the left side is the projective r-space over $\mathbb{C}$ viewed as a complex manifold (or a complex analytic variety, on which more later). Where does this morphism come from? We'll give a functorial answer later, but for now I'll do something more direct.

For each $i = 0, \dots, r$, put $X_i = D_+(x_i) \subseteq \mathbb{P}^r_{\mathbb{C}}$. This space is an affine n-space over $\mathbb{C}$ with coordinates x_j/x_i for $j \neq i$; let $\tilde{X}_i$ be the complex analytic affine r-space with the same coordinates. There is an obvious map

$$\Gamma(X_i, \mathcal{O}_{X_i}) = \mathbb{C}[x_0/x_i, \dots, x_r/x_i] \to \Gamma(\tilde{X}_i, \mathcal{O}_{\tilde{X}_i});$$

by adjunction, this gives us a morphism

$$\tilde{X}_i \to X_i$$

of locally ringed spaces. These glue to give the morphism I described. Note that $\tilde{\mathbb{P}}^r_{\mathbb{C}}$ includes only some of the points of $\mathbb{P}^r_{\mathbb{C}}$ (namely the closed points), but gives them a finer topology (the analytic topology rather than the Zariski topology). This is consistent with the fact that the map $\tilde{\mathbb{P}}^r_{\mathbb{C}} \to \mathbb{P}^r_{\mathbb{C}}$ is continuous.

What is nice about viewing the analytification process this way is that we can apply operations defined on locally ringed spaces uniformly to both $\tilde{\mathbb{P}}^r_{\mathbb{C}}$ to $\mathbb{P}^r_{\mathbb{C}}$. For instance, for any quasicoherent sheaf $\mathcal{F}$ on $\mathbb{P}^r_{\mathbb{C}}$, the pullback $h^*\mathcal{F}$ is a quasicoherent sheaf on $\tilde{\mathbb{P}}^r_{\mathbb{C}}$.

Lemma (Cartan)**.** *For any* coherent *sheaf $\mathcal{F}$ on $\mathbb{P}^r_{\mathbb{C}}$, $h^*\mathcal{F}$ is coherent.*

Proof. Recall that there exists a surjection $\mathcal{O}(n)^{\oplus m} \to \mathcal{F}$ for some integers m, n. It thus suffices to show that $h^*\mathcal{O}(n)$ is coherent. Since coherence is a local property, it is enough to show that the structure sheaf on complex analytic affine n-space is coherent (as a module over itself). This follows from the fact that each local ring of this space is *noetherian*.

I won't give a complete proof of this here, but the basic idea is as follows. Let $\mathbb{C}\{x_1, \dots, x_r\}$ be the ring of formal power series which converge in some neighborhood of the origin; this is the ring we are trying to prove is noetherian. We proceed by induction on r, the case $r = 0$ being trivial. The key to the induction step is the *Weierstrass preparation theorem*, which implies that any element of $\mathbb{C}\{x_1, \dots, x_r\}$ equals a unit times an element of $\mathbb{C}\{x_1, \dots, x_{r-1}\}[x_r]$. Since that ring is noetherian by the induction hypothesis plus the Hilbert basis theorem, we deduce that $\mathbb{C}\{x_1, \dots, x_r\}$ is too. For a proof of the Weierstrass preparation theorem, see for example the first few pages of Griffiths and Harris, *Principles of Algebraic Geometry.* $\square$

We also need the following relationship between analytic and algebraic stalks.

Lemma. *For any $z \in \tilde{\mathbb{P}}^r_{\mathbb{C}}$, the morphism $f : \mathcal{O}_{\mathbb{P}^r_{\mathbb{C}},z} \to \mathcal{O}_{\tilde{\mathbb{P}}^r_{\mathbb{C}},z}$ is flat. That is, the morphism $h : \tilde{\mathbb{P}}^r_{\mathbb{C}} \to \mathbb{P}^r_{\mathbb{C}}$ is flat.*

Proof. Let $t_1, \dots, t_n$ be local (algebraic) coordinates at z. Then we have a completion morphism $g : \mathcal{O}_{\tilde{\mathbb{P}}^r_{\mathbb{C}},z} \to \mathbb{C}[\![t_1, \dots, t_n]\!]$. Both g and $g \circ f$ are faithfully flat because they are maps from noetherian local rings into their completions for their maximal ideals. This easily yields flatness of f. $\square$

Corollary. *The functor h^* from quasicoherent sheaves on $\mathbb{P}^r_{\mathbb{C}}$ to quasicoherent sheaves on $\tilde{\mathbb{P}}^r_{\mathbb{C}}$ is exact.*

3 The first GAGA theorem

Note that for any quasicoherent sheaf $\mathcal{F}$ on $\mathbb{P}^r_{\mathbb{C}}$, there is always a natural morphism

$$H^i(\mathbb{P}^r_{\mathbb{C}}, \mathcal{F}) \to H^i(\tilde{\mathbb{P}}^r_{\mathbb{C}}, h^*\mathcal{F})$$

by pulling back along h. More concretely, you may view an algebraic Čech cocycle as an analytic one.

Theorem (GAGA, part 1)**.** *For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r_{\mathbb{C}}$, the natural morphism*

$$H^i(\mathbb{P}^r_{\mathbb{C}}, \mathcal{F}) \to H^i(\tilde{\mathbb{P}}^r_{\mathbb{C}}, h^*\mathcal{F})$$

is an isomorphism for each $i \geq 0$.

In order to prove this, we need a mechanism for computing sheaf cohomology on analytic spaces. Here it is, presented as a black box.

Theorem (Cartan)**.** *For any nonempty subset J of $\{0, \dots, r\}$ and any coherent sheaf $\mathcal{F}$ on $U = \cap_{j \in J} \tilde{X}_j$, $H^i(U, \mathcal{F}) = 0$ for $i > 0$.*

The key point is that U is a *Stein manifold.* This also holds if U is the analytification of any affine scheme of finite type over $\mathbb{C}$ (which I'll leave to you to define). By Leray's theorem, this gives the following corollary.

Corollary. *For any coherent sheaf $\mathcal{F}$ on $\tilde{\mathbb{P}}^r_{\mathbb{C}}$, we may compute sheaf cohomology using the Čech complex associated to the cover $\mathfrak{U} = \{X_0, \ldots, X_r\}$. In particular, the i-th cohomology vanishes for $i > r$.*

With this, the proof is parallel to that of Serre's finiteness theorem.

Proof of GAGA (part 1). We first prove the claim for $\mathcal{F} = \mathcal{O}$ by an explicit Čech cohomology calculation (exercise); note that the computation $H^0(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{O}) = \mathbb{C}$ comes down to the fact that any bounded entire function on $\mathbb{C}^n$ is constant, which reduces to Liouville's theorem. (By the way, this makes it clear that the theorem is completely false if we replace $\mathbb{P}^r_{\mathbb{C}}$ with, say, the affine space $\mathbb{A}^r_{\mathbb{C}}$. More on this later.)

We next deal with the cases $\mathcal{F} = \mathcal{O}(n)$ for $n \in \mathbb{Z}$, using the exact sequence

$$0 \to \mathcal{O}(n-1) \stackrel{\times x_r}{\to} \mathcal{O}(n) \to \mathcal{O}_H(n) \to 0$$

for $H \cong \mathbb{P}^{r-1}_{\mathbb{C}}$ the hyperplane $x_r = 0$. By induction on r, and comparing long exact sequences in cohomology, we can infer all of the cases from the case $n = 0$.

Finally, we treat the general case by descending induction on i (as in the proof that Čech cohomology for a good cover computes sheaf cohomology). Build an exact sequence

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$$

in which $\mathcal{E}$ is a direct sum of twisting sheaves. Note that applying h^* is exact, so we get an exact sequence on the analytic side. Then twist and compare long exact sequences in cohomology after twisting:

$$\begin{array}{ccccccccc}
\cdots \to & H^i(\mathbb{P}^r_{\mathbb{C}}, \mathcal{E}(n)) & \to & H^i(\mathbb{P}^r_{\mathbb{C}}, \mathcal{F}(n)) & \to & H^{i+1}(\mathbb{P}^r_{\mathbb{C}}, \mathcal{G}(n)) & \to & H^{i+1}(\mathbb{P}^r_{\mathbb{C}}, \mathcal{E}(n)) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & H^i(\tilde{\mathbb{P}}^r_{\mathbb{C}}, h^*\mathcal{E}(n)) & \to & H^i(\tilde{\mathbb{P}}^r_{\mathbb{C}}, h^*\mathcal{F}(n)) & \to & H^{i+1}(\tilde{\mathbb{P}}^r_{\mathbb{C}}, h^*\mathcal{G}(n)) & \to & H^{i+1}(\tilde{\mathbb{P}}^r_{\mathbb{C}}, h^*\mathcal{E}(n)) & \to \cdots
\end{array}$$

Using the five lemma, we get the desired result. □

4 The second GAGA theorem

We now know that algebraic coherent sheaves preserve their cohomology under pullback to the analytic side. We next show that they also preserve their morphisms.

Theorem (GAGA, part 2)**.** *Let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on $\mathbb{P}^r_{\mathbb{C}}$. Then the natural map*

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^r_{\mathbb{C}}}}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_{\tilde{\mathbb{P}}^r_{\mathbb{C}}}}(h^*\mathcal{F}, h^*\mathcal{G})$$

is an isomorphism.

Proof. In general, for sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, let $\mathscr{H}om(\mathcal{F},\mathcal{G})$ be the presheaf

$$\mathscr{H}om(\mathcal{F},\mathcal{G})(U) = \mathrm{Hom}_{O_U}(\mathcal{F}|_U, \mathcal{G}_U).$$

This is in fact a sheaf, called the *sheaf Hom* from $\mathcal{F}$ to $\mathcal{G}$. Its global sections are just $\mathrm{Hom}_{O_X}(\mathcal{F},\mathcal{G})$. (I should really write $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ with the subscript $\mathcal{O}_X$, but never mind for now.)

Note that there is a natural map

$$h^*\mathscr{H}om(\mathcal{F},\mathcal{G}) \to \mathscr{H}om(h^*\mathcal{F}, h^*\mathcal{G})$$

of sheaves on $\tilde{\mathbb{P}}^r_{\mathbb{C}}$, given by viewing an algebraic morphism over a Zariski open subset of $\mathbb{P}^r_{\mathbb{C}}$ as an analytic morphism over the corresponding subset of $\tilde{\mathbb{P}}^r_{\mathbb{C}}$. We claim this map is an isomorphism; this will imply the theorem by taking global sections of this isomorphism, then applying the first GAGA theorem.

We check the isomorphism on stalks. Using coherence, we have for each $z \in \tilde{\mathbb{P}}^r_{\mathbb{C}}$ a natural identification

$$\mathscr{H}om(\mathcal{F},\mathcal{G})_z = \mathrm{Hom}(\mathcal{F}_z, \mathcal{G}_z)$$

and similarly on the analytic side. Put

$$R = \mathcal{O}_{\mathbb{P}^r_{\mathbb{C}},z}, \qquad \tilde{R} = \mathcal{O}_{\tilde{\mathbb{P}}^r_{\mathbb{C}},z};$$

a lemma from earlier states that $\tilde{R}$ is flat over R. By that flatness plus the lemma below (and the fact that R is noetherian), we have a natural identification

$$\mathrm{Hom}(\mathcal{F}_z, \mathcal{G}_z) \otimes_R \tilde{R} = \mathrm{Hom}(\mathcal{F}_z \otimes_R \tilde{R}, \mathcal{G}_z \otimes_R \tilde{R}).$$

This yields the claim. □

Lemma. *Let R be a noetherian ring. Let S be a flat R-algebra. Then for any R-modules M, N, the natural map*

$$\mathrm{Hom}_R(M,N) \otimes_R S \to \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$$

is a bijection.

Proof. Since R is noetherian, I can find an exact sequence

$$F_1 \to F_0 \to M \to 0$$

where F_0, F_1 are *finite* free R-modules. Then we get a diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_R(F_1,N) \otimes_R S & \longrightarrow & \mathrm{Hom}_R(F_0,N) \otimes_R S & \longrightarrow & \mathrm{Hom}_R(M,N) \otimes_R S \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \mathrm{Hom}_S(F_1 \otimes_R S, N \otimes_R S) & \longrightarrow & \mathrm{Hom}_S(F_0 \otimes_R S, N \otimes_R S) & \longrightarrow & \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)
\end{array}$$

with exact rows (the exactness in the first row requiring the flatness of S over R). Since the second and third vertical arrows are isomorphisms, so is the first by the five lemma. □

5 The third GAGA theorem

We next try to classify the coherent sheaves on the analytic projective space. We need one more black box.

Theorem (Cartan, Serre)**.** *For $\mathcal{F}$ a coherent sheaf on $\tilde{\mathbb{P}}^r_{\mathbb{C}}$, the spaces $H^i(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{F})$ are finite dimensional over $\mathbb{C}$ for all $i \geq 0$.*

Sketch of proof. Equip the Čech cocycles for the usual open cover with the topology of uniform convergence on compact subsets. Then restrict to a cover in which each $\tilde{X}_i$ is replaced by an open subset with closure inside $\tilde{X}_i$. Using this, one sees that for the induced topology on $H^i(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{F})$, the identity map is *compact*, which is only possible if this vector space is finite dimensional over $\mathbb{C}$. □

Theorem (GAGA, part 3)**.** *Every coherent sheaf on $\tilde{\mathbb{P}}^r_{\mathbb{C}}$ is the pullback under h of a unique coherent sheaf on $\mathbb{P}^r_{\mathbb{C}}$.*

The uniqueness follows from the second GAGA theorem. To prove existence, we induct on r, the case $r = 0$ being trivial. For $\mathcal{F}$ a coherent sheaf on $\tilde{\mathbb{P}}^r_{\mathbb{C}}$, we extend the twisting notation from the algebraic case by writing

$$\mathcal{F}(n) = \mathcal{F} \otimes h^*\mathcal{O}(n).$$

Lemma. *Assume the third GAGA theorem in dimensions up to $r - 1$. For any coherent sheaf $\mathcal{F}$ on $\tilde{\mathbb{P}}^r_{\mathbb{C}}$ and any $z \in \tilde{\mathbb{P}}^r_{\mathbb{C}}$, there exists an integer n_0 (depending on $\mathcal{F}$ and z) such that for $n \geq n_0$, $\mathcal{F}(n)_z$ is generated by global sections of $\mathcal{F}(n)$.*

Proof. Choose $x_r \in H^0(\mathbb{P}^r_{\mathbb{C}}, \mathcal{O}(1))$ vanishing at z, and let E be the hyperplane $x_r = 0$. We then have the usual exact sequence

$$0 \to \mathcal{O}(-1) \stackrel{\times x_r}{\to} \mathcal{O} \to \mathcal{O}_E \to 0$$

of algebraic coherent sheaves. Tensoring with $\mathcal{F}$, we have an exact sequence

$$\mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_E \to 0$$

where $\mathcal{F}_E$ denotes the pushforward of the restriction to E. Let $\mathcal{G}$ be the kernel on the left side. Twisting, we get

$$0 \to \mathcal{G}(n) \to \mathcal{F}(n-1) \to \mathcal{F}(n) \to \mathcal{F}_E(n) \to 0.$$

Split this into short exact sequences:

$$0 \to \mathcal{G}(n) \to \mathcal{F}(n-1) \to \mathcal{H} \to 0$$
$$0 \to \mathcal{H} \to \mathcal{F}(n) \to \mathcal{F}_E(n) \to 0$$

and then take long exact sequences in cohomology:

$$H^1(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{F}(n-1)) \to H^1(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{H}) \to H^2(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{G}(n))$$
$$H^1(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{H}) \to H^1(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{F}(n)) \to H^1(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{F}_E(n)).$$

Note that $\mathcal{G}$ and $\mathcal{F}_E$ are supported on E, so by the induction hypothesis, they both come from algebraic coherent sheaves. It follows that for n large enough, the terms $H^2(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{G}(n))$ and $H^1(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{F}_E(n))$ both vanish. We thus obtain inequalities

$$\dim_{\mathbb{C}} H^1(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{F}(n-1)) \geq \dim_{\mathbb{C}} H^1(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{H}) \geq \dim_{\mathbb{C}} H^1(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{F}(n))$$

for n large. By the previous Cartan theorem, the terms of the sequence $\dim_{\mathbb{C}} H^1(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{F}(n))$ are all finite; we just showed that they are nonincreasing for n large enough. They thus eventually reach a *constant value* for n large enough!

In particular, for n large, the previous inequalities all become equalities. Backing up the second of the two long exact sequences, we see that

$$H^0(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{F}(n)) \to H^0(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{F}_E(n))$$

must be surjective for n large.

Again since $\mathcal{F}_E$ is known to be algebraic, for n large, $H^0(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{F}_E(n))$ generates $(\mathcal{F}_E)_z$. By a quick Nakayama's lemma argument, for such n, $H^0(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{F}(n))$ also generates $\mathcal{F}(n)_z$. □

Corollary. *Assume the third GAGA theorem in dimensions up to $r-1$. For any coherent sheaf $\mathcal{F}$ on $\tilde{\mathbb{P}}^r_{\mathbb{C}}$, there exists an integer n_0 (depending only on $\mathcal{F}$) such that for any $n \geq n_0$ and any $z \in \tilde{\mathbb{P}}^r_{\mathbb{C}}$, $\mathcal{F}(n)_z$ is generated by global sections of $\mathcal{F}(n)$.*

Proof. For a single n, if the claim holds for a single z, it holds in a neighborhood of that z; moreover, by multiplying these sections by monomials in $x_0, \dots, x_r$, we infer the claim for all larger n *in the same neighborhood*. Since $\tilde{\mathbb{P}}^r_{\mathbb{C}}$ is compact, we may find a single n_0 such that $\mathcal{F}(n)_z$ is generated by global sections of $\mathcal{F}(n)$ for each $z \in \tilde{\mathbb{P}}^r_{\mathbb{C}}$ and each $n \geq n_0$. □

Proof of the theorem. Let $\mathcal{F}$ be a coherent sheaf on $\tilde{\mathbb{P}}^r_{\mathbb{C}}$. By the previous corollary, for some n, each stalk of $\mathcal{F}(n)$ is generated by the space of sections $H^0(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{F}(n))$, which by Cartan's theorem is finite dimensional over $\mathbb{C}$. We thus obtain a surjection $h^*\mathcal{O}(-n)^{\oplus m} \to \mathcal{F}$ for some m, n. Applying the same argument to the kernel of this map (which is again coherent), we get an exact sequence

$$\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F} \to 0$$

in which each $\mathcal{F}_i$ is a direct sum of pullbacks of twisting sheaves. In particular, the $\mathcal{F}_i$ are algebraic; by the second GAGA theorem, the morphism between them is also algebraic. We may then form the algebraic cokernel, whose analytification is isomorphic to $\mathcal{F}$, as desired. □

6 More analytification

One can state the GAGA theorems more generally, but first we need to discuss analytification of spaces other than projective space. We first specify the target category: a locally ringed space $(X, \mathcal{O}_X)$ is a *complex analytic space* if each point x admits a neighborhood U and an immersion $\phi : U \to \mathbb{C}^n$ for some n. This is not the same as a *complex manifold* because we allow singularities, and for that matter nonreducedness (so these shouldn't be called *complex analytic varieties* either). Let $\underline{\text{AnSp}}$ denote the category of complex analytic spaces.

We would like a process for turning schemes locally of finite type over $\mathbb{C}$ into complex analytic spaces in a natural way. It is easy to say what we want to have happen in local coordinates: if $X = \operatorname{Spec} \mathbb{C}[z_1, \dots, z_n]/(f_1, \dots, f_m)$, we want to take the subspace Z of $\mathbb{C}^n$ on which $f_1 = \cdots = f_m = 0$, equipped with the quotient of $\mathcal{O}_{\mathbb{C}^n}$ by the coherent ideal sheaf generated by $f_1, \dots, f_m$ (or rather, its inverse image on Z).

However, if one works this way, one has to check independence from coordinates. This is doable but annoying (it's like Hartshorne's Proposition II.2.14 comparing certain schemes to varieties). There is a more functorial description of analytification introduced by Grothendieck; see SGA I, exposé XII, Théorème-Définition 1.1.

Theorem. *Let X be a scheme locally of finite type over $\mathbb{C}$. The functor*

$$Y \mapsto \operatorname{Hom}_{\underline{\text{LocRingSp}}}(Y, X)$$

from $\underline{\text{AnSp}}$ to $\underline{\text{Set}}$ is represented by an analytic space X^{an}; that is, there are natural isomorphisms

$$\operatorname{Hom}_{\underline{\text{LocRingSp}}}(Y, X) \to \operatorname{Hom}_{\underline{\text{AnSp}}}(Y, X^{\text{an}}).$$

Moreover, X^{an} has underlying set $X(\mathbb{C})$, and the morphism $X^{\text{an}} \to X$ induces isomorphisms of completed local rings, and so is flat.

You could interpret this as saying that the inclusion functor from analytic spaces to locally ringed spaces has a "partial right adjoint".

Sketch of proof. One first shows that the class of schemes for which the theorem holds is closed under forming open subschemes, closed subschemes, and products, by mirroring these constructions on the analytic side. It then suffices to check the theorem for $X = \mathbb{A}^1_{\mathbb{C}}$; this amounts to observing that giving a map $Y \to X$ is the same (by the adjunction property of affine schemes) as specifying a map $\mathbb{C}[t] \to \Gamma(Y, \mathcal{O}_Y)$, which in turn is the same as specifying the image of t. That is, $\operatorname{Hom}(Y, X)$ is naturally isomorphic to $\Gamma(Y, \mathcal{O}_Y)$. On the other hand, if we view $\mathbb{C}$ as an analytic space in the obvious fashion, then we may again identify $\operatorname{Hom}(Y, \mathbb{C})$ naturally with holomorphic functions on Y, i.e., with $\Gamma(Y, \mathcal{O}_Y)$. This proves the claim for affine space. $\square$

This paradigm extends to other categories derived from schemes. For instance, for k an algebraically closed field, separated reduced schemes of finite type over k admit "varietifications", thus reproducing the class of abstract algebraic varieties and giving a stronger version of Hartshorne Proposition II.2.14.

7 Extension to projective and proper schemes

In terms of the analytification functor, we can now extend the GAGA theorems as follows.

Theorem (GAGA for projective schemes)**.** *Let X be a closed subscheme of $\mathbb{P}^r_{\mathbb{C}}$ for some $r \geq 1$. Let $h : X^{\mathrm{an}} \to X$ be the analytification morphism.*

(a) For any coherent sheaf $\mathcal{F}$ on X, the natural morphism

$$H^i(X, \mathcal{F}) \to H^i(X^{\mathrm{an}}, h^*\mathcal{F})$$

is an isomorphism.

(b) For any coherent sheaves $\mathcal{F}, \mathcal{G}$ on X, the natural morphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_{X^{\mathrm{an}}}}(h^*\mathcal{F}, h^*\mathcal{G})$$

is an isomorphism.

(c) Every coherent sheaf on X^{an} is isomorphic to $h^*\mathcal{F}$ for a unique coherent sheaf $\mathcal{F}$ on X.

We saw earlier that already (a) is totally false for $X = \mathbb{A}^r_{\mathbb{C}}$, so some sort of completeness is necessary. Grothendieck noticed that it suffices to assume X is *proper* over $\mathbb{C}$; this reduces to the projective case using Chow's lemma (exercise).

8 Applications

The GAGA theorem has applications too numerous to count, so I'll just mention a few (see SGA 1, exposé XII for more). The following was proved before GAGA by Chow, but is an immediate corollary.

Corollary (Chow)**.** *Any closed analytic subvariety of $\tilde{\mathbb{P}}^r_{\mathbb{C}}$ is the analytification of a closed algebraic subvariety.*

Another application is the following.

Theorem. *Let X be a smooth proper scheme over $\mathbb{C}$. Then*

$$H^p(X, \Omega^q_{X/\mathbb{C}}) = H^p(X^{\mathrm{an}}, \Omega^q_{X^{\mathrm{an}}}) \qquad (p, q \geq 0).$$

This can be used to show that the hypercohomology of the algebraic de Rham complex $\Omega^{\cdot}_{X/\mathbb{C}}$ coincides with the hypercohomology of the analytic de Rham complex. (If $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a left exact additive functor of abelian categories with $\mathcal{C}_1$ having enough injectives, the *hypercohomology* of a complex $C^{\cdot}$ is defined by forming a quasi-isomorphism $C^{\cdot} \to I^{\cdot}$ with the $I^{\cdot}$ all injective, and taking $h^i(F(I^{\cdot}))$. More on this construction a bit later.) This in turn can be combined with some more results on the analytic/topological side (the Dolbeault and de Rham theorems, respectively) to show that algebraic de Rham cohomology computes the usual topological Betti numbers of a smooth variety over $\mathbb{C}$.

Here is another application by Grothendieck. See SGA 1, exposé XII again.

Theorem (Grothendieck)**.** *Let X be a smooth proper scheme over $\mathbb{C}$. Then any finite covering space map $Y \to X^{\mathrm{an}}$ (of topological spaces) corresponds to a finite étale cover of X in the category of schemes.*

One can define the *étale fundamental group* of a scheme X as, roughly speaking, the automorphism group of a maximal inverse system of connected finite étale covers of X. For instance, if $X = \operatorname{Spec} F$ with F a field, this gives the absolute Galois group of F. (To make this more precise, one must fix a choice of a *basepoint* just as in the topological case.) The previous theorem implies that for a smooth proper scheme over $\mathbb{C}$, the étale fundamental group is the *profinite completion* of the usual topological fundamental group, i.e., the inverse limit of its finite quotients. For instance, for an elliptic curve, the topological fundamental group is $\mathbb{Z} \times \mathbb{Z}$, while the profinite completion is

$$\widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}} \cong \prod_p (\mathbb{Z}_p \times \mathbb{Z}_p),$$

where $\mathbb{Z}_p$ denotes the p-adic integers.

Corollary. *Let K be a number field and let X be a smooth proper scheme over K. Then the profinite completion of the fundamental group of $(X \times_K \mathbb{C})^{\mathrm{an}}$ does not depend on the choice of the embedding $K \hookrightarrow \mathbb{C}$.*

This might not be so surprising until I tell you that Serre exhibited examples in which the topological fundamental group *does* depend on the choice of the embedding! (Serre's example is a rather artificial construction using elliptic curves with complex multiplications. There are some more natural examples due to one of our postdocs, Junecue Suh.)

The following is an example of a rather large class of results from SGA 1. See the exercises for an example involving properness.

Theorem. *Let $f : X \to Y$ be a morphism of schemes locally of finite type over $\mathbb{C}$. Then f is separated if and only if $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is separated. In particular, X is separated if and only if X^{an} is Hausdorff.*

9 Analogues

I know of at least two analogues of GAGA, though there may be more.

- One is *formal GAGA*, in which one passes from a scheme to its formal completion along a closed subscheme.

- The other is *rigid GAGA*, which is like ordinary GAGA except that one works over a *complete nonarchimedean field*, and uses Tate's notion of *rigid analytic geometry* (or Berkovich's notion of *nonarchimedean analytic geometry* instead of complex analytic geometry.

- I suppose there is also an instance of GAGA for passing from reduced separate schemes of finite type over an algebraically closed field to abstract algebraic varieties. But that one is neither surprising nor useful.