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18.726 Algebraic Geometry
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18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Cohen-Macaulay schemes and Serre duality

In this lecture, we extend Serre duality to Cohen-Macaulay schemes over a field. As in the previous lecture, let k be a field (not necessarily algebraically closed), let $j : X \rightarrow P = \mathbb{P}_k^N$ be a closed immersion with X of dimension n , and let $\mathcal{O}_X(1)$ be the corresponding twisting sheaf.

1 Cohen-Macaulay schemes and duality

Let ω_X° denote a dualizing sheaf on X ; remember that this choice includes a trace map $H^n(X, \omega_X^\circ) \rightarrow k$. We then obtain natural functorial maps

$$\theta^i : \text{Ext}_X^i(\mathcal{F}, \omega_X^\circ) \rightarrow H^{n-i}(X, \mathcal{F})^\vee$$

because both sides are cohomological functors on the opposite category of coherent sheaves on X , and the one on the left is effaceable because it vanishes on direct sums of twisting sheaves. By the definition of a dualizing sheaf, θ^0 is always an isomorphism.

Theorem. *The following conditions are equivalent.*

- (a) *The scheme X is equidimensional (each irreducible component has dimension n) and Cohen-Macaulay.*
- (b) *The maps θ^i are isomorphisms for all $i \geq 0$ and all coherent sheaves \mathcal{F} on X .*

This is of course meaningless if I don't tell you what a Cohen-Macaulay scheme is. For the moment, suffice to say that a scheme is Cohen-Macaulay if and only if each of its local rings is a Cohen-Macaulay ring. That already has content, because then the theorem says that (b) is equivalent to a local condition on X , which is far from obvious.

I'll also point out that a regular local ring is always Cohen-Macaulay. This implies the following.

Corollary. *If X is smooth over k , then θ^i is an isomorphism for all $i \geq 0$ and all coherent sheaves \mathcal{F} on X .*

2 Proof of the duality theorem, part 1

Even without knowing what a Cohen-Macaulay scheme is, we can at least start working to prove that condition (b) is equivalent to a *local* condition on X . Let us start by relating (b) to two global vanishing assertions.

Lemma. *The following conditions are equivalent to (b).*

(c) For any locally free coherent sheaf \mathcal{F} on X , for q sufficiently large, we have $H^i(X, \mathcal{F}(-q)) = 0$ for all $i < n$.

(c') For q sufficiently large, we have $H^i(X, \mathcal{O}_X(-q)) = 0$ for all $i < n$.

Note that condition (c) is a sort of opposite to Serre's vanishing theorem, which gives the vanishing of $H^i(X, \mathcal{F}(q))$ for $i > 0$ and q sufficiently large.

Proof. Given (b), for any locally free coherent sheaf \mathcal{F} on X , we have

$$\begin{aligned} H^i(X, \mathcal{F}(-q)) &= \text{Ext}_X^{n-i}(\mathcal{F}(-q), \omega_X^\circ)^\vee \\ &= \text{Ext}_X^{n-i}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X^\circ(q))^\vee \\ &= H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ(q))^\vee \end{aligned}$$

and this vanishes for $n - i > 0$ and q large by Serre's vanishing theorem. Thus (b) implies (c).

It is clear that (c) implies (c'). Given (c'), it follows that $H^{n-i}(X, \cdot)^\vee$ is effaceable for all $i > 0$ since we can cover \mathcal{F} with a direct sum of twisting sheaves. Hence θ^i is the natural map between two universal cohomological functors, hence is an isomorphism. Thus (c') implies (b). \square

We next reformulate this in local terms, using Serre duality on P .

Lemma. *The following condition is equivalent to (b).*

(d) For all $i < n$, $\mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P) = 0$.

Remember that no matter what X is, we have $\mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P) = 0$ for $i > n$: we proved this in the course of constructing the dualizing sheaf ω_X° .

Proof. By Serre duality on P (and choosing an isomorphism $H^n(P, \omega_P) \cong k$), we may identify

$$H^i(X, \mathcal{O}_X(-q)) \cong H^i(P, j_*\mathcal{O}_X(-q)) \cong \text{Ext}_P^{N-i}(j_*\mathcal{O}_X, \omega_P(q))^\vee.$$

So (c) is equivalent to the condition that for q sufficiently large, $\text{Ext}_P^{N-i}(j_*\mathcal{O}_X, \omega_P(q)) = 0$ for all $i < n$. Recall from earlier that for q large,

$$\text{Ext}_P^{N-i}(j_*\mathcal{O}_X, \omega_P(q)) = \Gamma(P, \mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P(q))) = \Gamma(P, \mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P)(q)).$$

Since $\mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P)$ is coherent, $\Gamma(P, \mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P)(q))$ vanishes for q sufficiently large if and only if $\mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P) = 0$. \square

Condition (d) can be rewritten as follows.

Lemma. *The following condition is equivalent to (b).*

(e) For each point $x \in X$, if $A = \mathcal{O}_{P,x}$ and I is the ideal of A defining X at x , then for all $i < n$, $\text{Ext}_A^{N-i}(A/I, A) = 0$.

Proof. This translates directly from (d) once we remember that ω_P is locally free of rank 1 on P . \square

This is almost the local condition we are seeking, except that it still refers to the position of X within P .

3 The Cohen-Macaulay condition

To get rid of the dependence of our duality condition on the relative geometry of X within P , we need some more sophisticated commutative algebra.

Proposition. *Let A be a regular local ring and let M be a finitely generated A -module. Then for any nonnegative integer n , the following are equivalent.*

- (a) *We have $\text{Ext}^i(M, A) = 0$ for all $i > n$.*
- (b) *For any A -module N , we have $\text{Ext}^i(M, N) = 0$ for all $i > n$.*
- (c) *There exists a projective resolution $0 \rightarrow L_n \rightarrow \cdots \rightarrow L_1 \rightarrow L_0 \rightarrow M \rightarrow 0$ of M at length at most n .*

Proof. See Hartshorne Proposition III.6.10A (and associated Matsumura reference) and exercise III.6.6. \square

The smallest integer for which this holds is called the *projective dimension* of M (if it exists), denoted $\text{pd}_A(M)$. For instance, M is projective if and only if $\text{pd}_A(M) = 0$.

For M a module over a ring A , a *regular sequence* is a sequence x_1, \dots, x_n of elements of A such that for $i = 1, \dots, n$, x_i is not a zerodivisor on $M/(x_1, \dots, x_{i-1})M$. For A a local ring, the *depth* of M is the maximal length of a regular sequence with all x_i in the maximal ideal of A .

Proposition. *For A a regular local ring and M an A -module,*

$$\text{pd}_A(M) + \text{depth}_A(M) = \dim(A).$$

Proof. See Hartshorne Proposition III.6.12A (and associated Matsumura reference). \square

We can finally give a local equivalent to condition (b) from the duality theorem. Recall that our last equivalent (e) said that for each $x \in X$, for $A = \mathcal{O}_{P,x}$ and I the ideal of A defining X at x , $\text{Ext}_A^{N-i}(A/I, A) = 0$ for all $i < n$. This is equivalent to $\text{pd}_A(A/I) \leq N - n$, and hence to $\text{depth}_A(A/I) \geq n$. The trick is that if M is an A/I -module, then $\text{depth}_A(M) = \text{depth}_{A/I}(M)$. Thus we have the following.

Lemma. *The following condition is equivalent to (b).*

- (f) *For each point $x \in X$, if $B = \mathcal{O}_{X,x}$, then $\text{depth}_B(B) \geq n$.*

On the other hand, we always have $\text{depth}_B(B) \leq \dim(B) \leq n$, so it is equivalent to require $\text{depth}_B(B) = \dim(B) = n$.

This condition $\text{depth}_B(B) = \dim(B)$ is in fact the definition of a *Cohen-Macaulay* local ring B . Any regular local ring is Cohen-Macaulay, since we can use generators of the cotangent space as a regular sequence. But the Cohen-Macaulay condition is much more permissive; for instance, any *local complete intersection* is Cohen-Macaulay.