SOLUTIONS TO EXERCISES

CHAPTER I

A. Manifolds

A.2. If p_1, $p_2 \in M$ are sufficiently close within a coordinate neighborhood U, there exists a diffeomorphism mapping p_1 to p_2 and leaving $M - U$ pointwise fixed. Now consider a curve segment $\gamma(t)$ $(0 \leqslant t \leqslant 1)$ in M joining p to q. Let t^* be the supremum of those t for which there exists a diffeomorphism of M mapping p on $\gamma(t)$. The initial remark shows first that $t^* > 0$, next that $t^* = 1$, and finally that t^* is reached as a maximum.

A.3. The "only if" is obvious and "if" follows from the uniqueness in Prop. 1.1. Now let $\mathfrak{F} = C^\infty(\boldsymbol{R})$ where $\boldsymbol{R}$ is given the ordinary differentiable structure. If n is an odd integer, let $\mathfrak{F}^n$ denote the set of functions $x \to f(x^n)$ on $\boldsymbol{R}$, $f \in \mathfrak{F}$ being arbitrary. Then $\mathfrak{F}^n$ satisfies $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_3$. Since $\mathfrak{F}^n \neq \mathfrak{F}^m$ for $n \neq m$, the corresponding δ^n are all different.

A.4. (i) If $d\Phi \cdot X = Y$ and $f \in C^\infty(N)$, then $X(f \circ \Phi) = (Yf) \circ \Phi \in \mathfrak{F}_0$. On the other hand, suppose $X\mathfrak{F}_0 \subset \mathfrak{F}_0$. If $F \in \mathfrak{F}_0$, then $F = g \circ \Phi$ where $g \in C^\infty(N)$ is unique. If $f \in C^\infty(N)$, then $X(f \circ \Phi) = g \circ \Phi$ ($g \in C^\infty(N)$ unique), and $f \to g$ is a derivation, giving Y.

(ii) If $d\Phi \cdot X = Y$, then $Y_{\Phi(p)} = d\Phi_p(X_p)$, so necessity follows. Suppose $d\Phi_p(M_p) = N_{\Phi(p)}$ for each $p \in M$. Define for $r \in N$, $Y_r = d\Phi_p(X_p)$ if $r = \Phi(p)$. In order to show that $Y : r \to Y_r$ is differentiable we use (by virtue of Theorem 15.5) coordinates around p and around $r = \Phi(p)$ such that Φ has the expression $(x_1, \ldots, x_m) \to (x_1, \ldots, x_n)$. Writing

$$X = \sum_1^m a_i(x_1, \ldots, x_m) \frac{\partial}{\partial x_i},$$

we have for q sufficiently near p

$$d\Phi_q(X_q) = \sum_1^n a_i(x_1(q), ..., x_m(q)) \left(\frac{\partial}{\partial x_i}\right)_{\Phi(q)},$$

so condition (1) implies that for $1 \leqslant i \leqslant n$, a_i is constant in the last $m - n$ arguments. Hence

$$Y = \sum_1^n a_i(x_1, ..., x_n, x_{n+1}(p), ..., x_m(p)) \frac{\partial}{\partial x_i}.$$

(iii) $f \in C^\infty(N)$ if and only if $f \circ \psi \in C^\infty(\boldsymbol{R})$. If $f(x) = x^3$, then $f \circ \psi(x) = x$, $(f' \circ \psi)(x) = 3x^{2/3}$, so $f \in C^\infty(N)$, $f' \notin C^\infty(N)$. Hence $f \circ \Phi \in \mathfrak{F}_0$, but $X(f \circ \Phi) \notin \mathfrak{F}_0$; so by (i), X is not projectable.

A.5. Obvious.

A.6. Use Props. 15.2 and 15.3 to shrink the given covering to a new one; then use the result of Exercise A.1 to imitate the proof of Theorem 1.3.

A.7. We can assume $M = \boldsymbol{R}^m$, $p = 0$, and that $X_0 = (\partial/\partial t_1)_0$ in terms of the standard coordinate system $\{t_1, ..., t_m\}$ on $\boldsymbol{R}^m$. Consider the integral curve $\varphi_t(0, c_2, ..., c_m)$ of X through $(0, c_2, ..., c_m)$. Then the mapping $\psi : (c_1, ..., c_m) \to \varphi_{c_1}(0, c_2, ..., c_m)$ is C^∞ for small c_i, $\psi(0, c_2, ..., c_m) = (0, c_2, ..., c_m)$, so

$$d\psi_0 \left(\frac{\partial}{\partial c_i}\right) = \left(\frac{\partial}{\partial t_i}\right)_0 \qquad (i > 1).$$

Also

$$d\psi_0 \left(\frac{\partial}{\partial c_1}\right)_0 = \left(\frac{\partial \varphi_{c_1}}{\partial c_1}\right)(0) = X_0 = \left(\frac{\partial}{\partial t_1}\right)_0.$$

Thus ψ can be inverted near 0, so $\{c_1, ..., c_m\}$ is a local coordinate system. Finally, if $c = (c_1, ..., c_m)$,

$$\begin{aligned}\left(\frac{\partial}{\partial c_1}\right)_{\psi(c)} f &= \left(\frac{\partial(f \circ \psi)}{\partial c_1}\right)_c \\ &= \lim_{h \to 0} \frac{1}{h} [f(\varphi_{c_1+h}(0, c_2, ..., c_m)) - f(\varphi_{c_1}(0, c_2, ..., c_m)] \\ &= (Xf)(\psi(c))\end{aligned}$$

so $X = \partial/\partial c_1$.

A.8. Let $f \in C^\infty(M)$. Writing $\sim$ below when in an equality we omit terms of higher order in s or t, we have

$$
\begin{aligned}
& f(\psi_{-t}(\varphi_{-s}(\psi_t(\varphi_s(o))))) - f(o) \\
&\quad = f(\psi_{-t}(\varphi_{-s}(\psi_t(\varphi_s(o))))) - f(\varphi_{-s}(\psi_t(\varphi_s(o)))) \\
&\qquad + f(\varphi_{-s}(\psi_t(\varphi_s(o)))) - f(\psi_t(\varphi_s(o))) \\
&\qquad + f(\psi_t(\varphi_s(o))) - f(\varphi_s(o)) + f(\varphi_s(o)) - f(o) \\
&\quad \sim -t(Yf)(\varphi_{-s}(\psi_t(\varphi_s(o)))) + \tfrac{1}{2}t^2(Y^2f)(\varphi_{-s}(\psi_t(\varphi_s(o)))) \\
&\qquad - s(Xf)(\psi_t(\varphi_s(o))) + \tfrac{1}{2}s^2(X^2f)(\psi_t(\varphi_s(o))) \\
&\qquad + t(Yf)(\psi_t(\varphi_s(o))) - \tfrac{1}{2}t^2(Y^2f)(\psi_t(\varphi_s(o))) \\
&\qquad + s(Xf)(\varphi_s(o)) - \tfrac{1}{2}s^2(X^2f)(\varphi_s(o)) \\
&\quad \sim st(XYf)(\psi_t(\varphi_s(o))) - st(YXf)(\psi_t(\varphi_s(o))).
\end{aligned}
$$

This last expression is obtained by pairing off the 1st and 5th term, the 3rd and 7th, the 2nd and 6th, and the 4th and 8th. Hence

$$f(\gamma(t^2)) - f(o) = t^2([X, Y]f)(o) + O(t^3).$$

A similar proof is given in Faber [1].

B. The Lie Derivative and the Interior Product

B.1. If the desired extension of $\theta(X)$ exists and if $C : \mathfrak{D}_1^1(M) \to C^\infty(M)$ is the contraction, then (i), (ii), (iii) imply

$$(\theta(X)\omega)(Y) = X(\omega(Y)) - \omega([X, Y]), \qquad X, Y \in \mathfrak{D}^1(M).$$

Thus we define $\theta(X)$ on $\mathfrak{D}_1(M)$ by this relation and note that $(\theta(X)\omega)(fY) = f(\theta(X)(\omega))(Y)$ $(f \in C^\infty(M))$, so $\theta(X)$ $\mathfrak{D}_1(M) \subset \mathfrak{D}_1(M)$. If U is a coordinate neighborhood with coordinates $\{x_1, \ldots, x_m\}$, $\theta(X)$ induces an endomorphism of $C^\infty(U)$, $\mathfrak{D}^1(U)$, and $\mathfrak{D}_1(U)$. Putting $X_i = \partial/\partial x_i$, $\omega_j = dx_j$, each $T \in \mathfrak{D}_s^r(U)$ can be written

$$T = \sum T_{(i),(j)} X_{i_1} \otimes \ldots \otimes X_{i_r} \otimes \omega_{j_1} \otimes \ldots \otimes \omega_{j_s}$$

with unique coefficients $T_{(i),(j)} \in C^\infty(U)$. Now $\theta(X)$ is uniquely extended to $\mathfrak{D}(U)$ satisfying (i) and (ii). Property (iii) is then verified by induction on r and s. Finally, $\theta(X)$ is defined on $\mathfrak{D}(M)$ by the condition $\theta(X)T \mid U = \theta(X)(T \mid U)$ (vertical bar denoting restriction) because as in the proof of Theorem 2.5 this condition is forced by the requirement that $\theta(X)$ should be a derivation.

B.2. The first part being obvious, we just verify $\Phi \cdot \omega = (\Phi^{-1})^*\omega$. We may assume $\omega \in \mathfrak{D}_1(M)$. If $X \in \mathfrak{D}^1(M)$ and C is the contraction $X \otimes \omega \to \omega(X)$, then $\Phi \circ C = C \circ \Phi$ implies $(\Phi \cdot \omega)(X) = \Phi(\omega(X^{\Phi^{-1}})) = ((\Phi^{-1})^*\omega)(X)$.

B.3. The formula is obvious if $T = f \in C^\infty(M)$. Next let $T = Y \in \mathfrak{D}^1(M)$. If $f \in C^\infty(M)$ and $q \in M$, we put $F(t, q) = f(g_t \cdot q)$ and have

$$F(t, q) - F(0, q) = t \int_0^1 \left(\frac{\partial F}{\partial t}\right)(st, q)\, ds = t\, h(t, q),$$

where $h \in C^\infty(\boldsymbol{R} \times M)$ and $h(0, q) = (Xf)(q)$. Then

$$(g_t \cdot Y)_p f = (Y(f \circ g_t))(g_t^{-1} \cdot p) = (Yf)(g_t^{-1} \cdot p) + t(Yh)(t, g_t^{-1} \cdot p)$$

so

$$\lim_{t \to 0} \frac{1}{t}(Y - g_t \cdot Y)_p f = (XYf)(p) - (YXf)(p),$$

so the formula holds for $T \in \mathfrak{D}^1(M)$. But the endomorphism $T \to \lim_{t\to 0} t^{-1}(T - g_t \cdot T)$ has properties (i), (ii), and (iii) of Exercise B.1; it coincides with $\theta(X)$ on $C^\infty(M)$ and on $\mathfrak{D}^1(M)$, hence on all of $\mathfrak{D}(M)$ by the uniqueness in Exercise B.1.

B.4. For (i) we note that both sides are derivations of $\mathfrak{D}(M)$ commuting with contractions, preserving type, and having the same effect on $\mathfrak{D}^1(M)$ and on $C^\infty(M)$. The argument of Exercise B.1 shows that they coincide on $\mathfrak{D}(M)$.

(ii) If $\omega \in \mathfrak{D}_r(M)$, $Y_1, \ldots, Y_r \in \mathfrak{D}^1(M)$, then by B.1,

$$(\theta(X)\omega)(Y_1, \ldots, Y_r) = X(\omega(Y_1, \ldots, Y_r)) - \sum_i \omega(Y_1, \ldots, [X, Y_i], \ldots, Y_r)$$

so $\theta(X)$ commutes with A.

(iii) Since $\theta(X)$ is a derivation of $\mathfrak{A}(M)$ and d is a *skew-derivation* (that is, satisfies (iv) in Theorem 2.5), the commutator $\theta(X)d - d\theta(X)$ is also a skew-derivation. Since it vanishes on f and df ($f \in C^\infty(M)$), it vanishes identically (cf. Exercise B.1). For B.1–B.4, cf. Palais [3].

B.5. This is done by the same method as in Exercise B.1.

B.6. For (i) we note that by (iii) in Exercise B.5, $i(X)^2$ is a derivation. Since it vanishes on $C^\infty(M)$ and $\mathfrak{D}_1(M)$, it vanishes identically; (ii) follows by induction; (iii) follows since both sides are skew-derivations which coincide on $C^\infty(M)$ and on $\mathfrak{A}_1(M)$; (iv) follows because both sides are derivations which coincide on $C^\infty(M)$ and on $\mathfrak{A}_1(M)$.

C. Affine Connections

C.1. M has a locally finite covering $\{U_\alpha\}_{\alpha \in A}$ by coordinate neighborhoods U_α. On U_α we construct an arbitrary Riemannian structure g_α. If $1 = \sum_\alpha \varphi_\alpha$ is a partition of unity subordinate to the covering, then $\sum_\alpha \varphi_\alpha g_\alpha$ gives the desired Riemannian structure on M.

C.2. If Φ is an affine transformation and we write $d\Phi(\partial/\partial x_j) = \sum_i a_{ij}\, \partial/\partial x_i$, then conditions ∇_1 and ∇_2 imply that each a_{ij} is a constant. If A is the linear transformation (a_{ij}), then $\Phi \circ A^{-1}$ has differential I, hence is a translation B, so $\Phi(X) = AX + B$. The converse is obvious.

C.3. We have $\Phi^*\omega_j^i = \sum_k (\Gamma_{kj}^i \circ \Phi)\, \Phi^*\omega^k$, so by (5′), (6), (7) in §8

$$\Phi^*\omega_j^i = \sum_k (\Gamma_{kj}^i \circ \Phi)(a_k\, dt + t\, da_k) = 0.$$

This implies that $\Gamma_{kj}^i \equiv 0$ in normal coordinates, which is equivalent to the result stated in the exercise.

C.4. A direct verification shows that the mapping $\delta : \theta \to \sum_1^m \omega_i \wedge \nabla_{X_i}(\theta)$ is a skew-derivation of $\mathfrak{A}(M)$ and that it coincides with d on $C^\infty(M)$. Next let $\theta \in \mathfrak{A}_1(M)$, $X, Y \in \mathfrak{D}^1(M)$. Then, using (5), §7,

$$\begin{aligned} 2\,\delta\theta(X, Y) &= 2 \sum_i (\omega_i \wedge \nabla_{X_i}(\theta))(X, Y) \\ &= \sum_i \omega_i(X)\, \nabla_{X_i}(\theta)(Y) - \omega_i(Y)\, \nabla_{X_i}(\theta)(X) \\ &= \nabla_X(\theta)(Y) - \nabla_Y(\theta)(X) \\ &= X \cdot \theta(Y) - \theta(\nabla_X(Y)) - Y \cdot \theta(X) + \theta(\nabla_Y(X)), \end{aligned}$$

which since the torsion is 0 equals

$$X\theta(Y) - Y \cdot \theta(X) - \theta([X, Y]) = 2\, d\theta(X, Y).$$

Thus $\delta = d$ on $\mathfrak{A}_1(M)$, hence by the above on all of $\mathfrak{A}(M)$.

C.5. No; an example is given by a circular cone with the vertex rounded off.

C.6. Using Props. 11.3 and 11.4 we obtain a mapping $\varphi : M \to N$ such that $d\varphi_p$ is an isometry for each $p \in M$. Thus $\varphi(M) \subset N$ is an open subset. Each geodesic in the manifold $\varphi(M)$ is indefinitely extendable, so $\varphi(M)$ is complete, whence φ maps M onto N. Now Lemma 13.4 implies that (M, φ) is a covering space of N, so M and N are isometric.

D. Submanifolds

D.1. Let $I : G_\Phi \to M \times N$ denote the identity mapping and $\pi : M \times N \to M$ the projection onto the first factor. Let $m \in M$ and $Z \in (G_\Phi)_{(m, \Phi(m))}$ such that $dI_m(Z) = 0$. Then $Z = (d\varphi)_m(X)$ where $X \in M_m$. Thus $d\pi \circ dI \circ d\varphi(X) = 0$. But since $\pi \circ I \circ \varphi$ is the identity mapping, this implies $X = 0$, so $Z = 0$ and I is regular.

D.2. Immediate from Lemma 14.1.

D.3. Consider the figure 8 given by the formula

$$\gamma(t) = (\sin 2t, \sin t) \qquad (0 \leqslant t \leqslant 2\pi).$$

Let $f(s)$ be an increasing function on $\boldsymbol{R}$ such that

$$\lim_{s\to-\infty} f(s) = 0, \qquad f(0) = \pi, \qquad \lim_{s\to+\infty} f(s) = 2\pi.$$

Then the map $s \to \gamma(f(s))$ is a bijection of $\boldsymbol{R}$ onto the figure 8. Carrying the manifold structure of $\boldsymbol{R}$ over, we get a submanifold of $\boldsymbol{R}^2$ which is closed, yet does not carry the induced topology. Replacing γ by δ given by $\delta(t) = (-\sin 2t, \text{sint } t)$, we get another manifold structure on the figure.

D.4. Suppose $\dim M < \dim N$. Using the notation of Prop. 3.2, let W be a compact neighborhood of p in M and $W \subset U$. By the countability assumption, countably many such W cover M. Thus by Lemma 3.1, Chapter II, for N, some such W contains an open set in N; contradiction.

D.5. For each $m \in M$ there exists by Prop. 3.2 an open neighborhood V_m of m in N and an extension of g from $V_m \cap M$ to a C^∞ function G_m on V_m. The covering $\{V_m\}_{m\in M}$, $N - M$ of N has a countable locally finite refinement $V_1, V_2, \ldots$. Let $\varphi_1, \varphi_2, \ldots$ be the corresponding partition of unity. Let $\varphi_{i_1}, \varphi_{i_2}, \ldots$ be the subsequence of the (φ_j) whose supports intersect M, and for each φ_{i_p} choose $m_p \in M$ such that $\operatorname{supp}(\varphi_{i_p}) \subset V_{m_p}$. Then $\Sigma_p\, G_{m_p}\, \varphi_{i_p}$ is the desired function G.

D.6. The "if" part is contained in Theorem 14.5 and the "only if" part is immediate from (2), Chapter V, §6.

E. Curvature

E.1. If (r, θ) are polar coordinates of a vector X in the tangent space M_p, the inverse of the map $(r, \theta) \to \operatorname{Exp}_p X$ gives the "geodesic polar coordinates" around p. Since the geodesics from p intersect sufficiently small circles around p orthogonally (Lemma 9.7), the Riemannian structure has the form $g = dr^2 + \varphi(r, \theta)^2\, d\theta^2$. In these coordinates the Riemannian measure $f \to \int f \sqrt{\bar{g}}\, dx_1 \ldots dx_n$ and the Laplace-Beltrami operator are, respectively, given by

$$f \to \iint f(r, \theta)\, \varphi(r, \theta)\, dr\, d\theta,$$

and

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \varphi^{-1} \frac{\partial \varphi}{\partial r} \frac{\partial f}{\partial r} + \varphi^{-1} \frac{\partial}{\partial \theta} \left(\varphi^{-1} \frac{\partial f}{\partial \theta}\right).$$

In particular

$$\Delta(\log r) = -\frac{1}{r^2} + \frac{1}{r\varphi}\frac{\partial\varphi}{\partial r}.$$

On the other hand, if (x, y) are the normal coordinates of $\mathrm{Exp}_p X$ such that

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x},$$

then, since $r\,dr = x\,dx + y\,dy$, $r^2\,d\theta = x\,dy - y\,dx$,

$$g = r^{-4}[(x^2r^2 + y^2\varphi^2)\,dx^2 + 2xy(r^2 - \varphi^2)\,dx\,dy + (y^2r^2 + x^2\varphi^2)\,dy^2]$$

so since the coefficients are smooth near $(x, y) = (0, 0)$ φ^2 has the form†

$$\varphi^2 = r^2 + cr^4 + \dots,$$

where $c = c(p)$ is a constant. But then

$$\lim_{r\to 0} \Delta(\log r) = c(p).$$

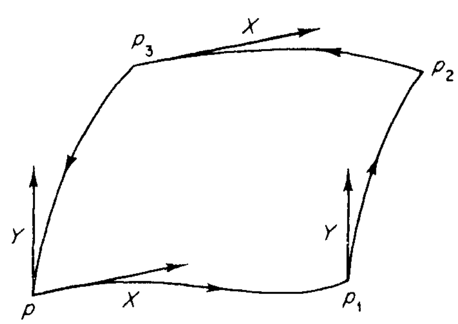
On the other hand,

$$A(r) = \int_0^r \int_0^{2\pi} \varphi(t, \theta)\,dt\,d\theta,$$

so using the definition in §12 we find $K = -3c(p)$ as stated.

This result is stated in Klein [1], p. 219, without proof (with opposite sign).

E.2. Let $X = \partial/\partial x_1$ and $Y = \partial/\partial x_2$ so γ_ϵ is formed by integral curves of X, Y, $-X$, $-Y$.



Let $p = p_0 = (0, 0, \dots, 0)$

$p_1 = (\epsilon, 0, \dots, 0)$

$p_2 = (\epsilon, \epsilon, \dots, 0)$

$p_3 = (0, \epsilon, \dots, 0)$

and τ_{ij} the parallel transport from p_j to p_i along γ_ϵ. Let T be any vector field on M, and write $T_i = T_{p_i}$. Then

$$\begin{aligned}\tau_{03}\tau_{32}\tau_{21}\tau_{10}T_0 - T_0 &= (\tau_{03}\tau_{32}\tau_{21}\tau_{10}T_0 - \tau_{03}\tau_{32}\tau_{21}T_1) + (\tau_{03}\tau_{32}\tau_{21}T_1 - \tau_{03}\tau_{32}T_2)\\ &\quad + (\tau_{03}\tau_{32}T_2 - \tau_{03}T_3) + (\tau_{03}T_3 - T_0).\end{aligned}$$

† See "Some Details," p. 586.

We use Theorem 7.1 and write $\sim$ when we omit terms of higher order in ϵ. Then our expression is

$$\sim \tau_{03}\tau_{32}\tau_{21}[-\epsilon(\nabla_X T)_1 + \tfrac{1}{2}\epsilon^2(\nabla_X^2 T)_1]$$
$$+ \tau_{03}\tau_{32}[-\epsilon(\nabla_Y T)_2 + \tfrac{1}{2}\epsilon^2(\nabla_Y^2 T)_2]$$
$$- \tau_{03}\tau_{32}[-\epsilon(\nabla_X T)_2 + \tfrac{1}{2}\epsilon^2(\nabla_X^2 T)_2]$$
$$- \tau_{03}[-\epsilon(\nabla_Y T)_3 + \tfrac{1}{2}\epsilon^2(\nabla_Y^2 T)_3].$$

Combining now the 1st and 5th term, 2nd and 6th term, etc., this expression reduces to

$$\sim \epsilon^2\tau_{03}\tau_{32}(\nabla_Y(\nabla_X(T)))_2 - \epsilon^2\tau_{03}(\nabla_X(\nabla_Y(T))_3$$

which, since $[X, Y] = 0$, reduces to

$$\sim \epsilon^2\tau_{03}(R(Y, X)T)_3 \sim \epsilon^2(R(Y, X)T)_0.$$

This proof is a simplification of that of Faber [1]. See Laugwitz [1], §10 for another version of the result. For curvature and holonomy groups, see e.g. Ambrose and Singer [2].

F. Surfaces

F.1. Let Z be a vector field on S and $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ vector fields on a neighborhood of s in $\boldsymbol{R}^3$ extending X, Y, and Z, respectively. The inner product $\langle\ ,\ \rangle$ on $\boldsymbol{R}^3$ induces a Riemannian structure g on S. If $\tilde{\nabla}$ and ∇ denote the corresponding affine connections on $\boldsymbol{R}^3$ and S, respectively, we deduce from (2), §9

$$\langle \tilde{Z}_s, \tilde{\nabla}_{\tilde{X}}(\tilde{Y})_s\rangle = g(Z_s, \nabla_X(Y)_s).$$

But

$$\tilde{\nabla}_{\tilde{X}}(\tilde{Y})_s = \lim_{t\to 0} \frac{1}{t}(Y_{\gamma(t)} - Y_s),$$

so we obtain $\nabla = \nabla'$; in particular ∇' is an affine connection on S.

F.2. Let $s(u, v) \to (u, v)$ be local coordinates on S and if g denotes the Riemannian structure on S, put

$$E = g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right), \qquad F = g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right), \qquad G = g\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right).$$

Let $r(u, v)$ denote the vector from 0 to the point $s(u, v)$. Subscripts denoting partial derivatives, r_u and r_v span the tangent space at $s(u, v)$, and we may take the orientation such that

$$\xi_{s(u,v)} = \frac{r_u \times r_v}{|r_u \times r_v|},$$

$\times$ denoting the cross product. We have

$$\dot{\gamma}_S = r_u \dot{u} + r_v \dot{v}$$
$$\ddot{\gamma}_S = r_{uu}\dot{u}^2 + 2r_{uv}\dot{u}\dot{v} + r_{vv}\dot{v}^2 + r_u\ddot{u} + r_v\ddot{v},$$

and

$$r_u \cdot r_u = E, \qquad r_u \cdot r_v = F, \qquad r_v \cdot r_v = G,$$

whence

$$r_{uu} \cdot r_u = \tfrac{1}{2}E_u, \qquad r_{uv} \cdot r_u = \tfrac{1}{2}E_v, \qquad r_{vv} \cdot r_v = \tfrac{1}{2}G_v,$$
$$r_{uv} \cdot r_v = \tfrac{1}{2}G_u, \qquad r_{uu} \cdot r_v = F_u - \tfrac{1}{2}E_v, \qquad r_{vv} \cdot r_u = F_v - \tfrac{1}{2}G_u.$$

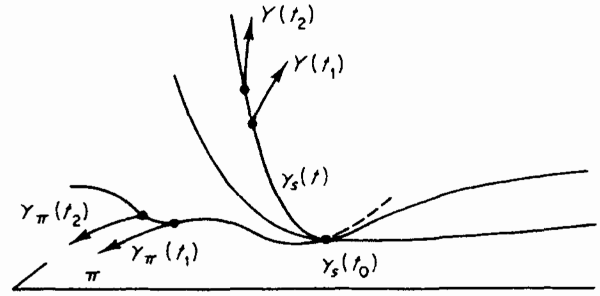
From this it is clear that the geodesic curvature can be expressed in terms of $\dot{u}$, $\dot{v}$, $\ddot{u}$, $\ddot{v}$, E, F, G, and their derivatives, and therefore has the invariance property stated.

F.3. We first recall that under the orthogonal projection P of R^3 on the tangent space $S_{\gamma_S(t)}$ the curve $P \circ \gamma_S$ has curvature in $\gamma_S(t)$ equal to the geodesic curvature of γ_S at $\gamma_S(t)$. So in order to avoid discussing developable surfaces we define the rolling in the problem as follows. Let $\pi = S_{\gamma_S(t_0)}$ and let $t \to \gamma_\pi(t)$ be the curve in π such that

$$\gamma_\pi(t_0) = \gamma_S(t_0), \qquad \dot{\gamma}_\pi(t_0) = \dot{\gamma}_S(t_0)$$

($t - t_0$ is the arc-parameter measured from $\gamma_\pi(t_0)$) and such that the curvature of γ_π at $\gamma_\pi(t)$ is the geodesic curvature of γ_S at $\gamma_S(t)$. The rolling is understood as the family of isometries $S_{\gamma_S(t)} \to \pi_{\gamma_\pi(t)}$ of the tangent planes such that the vector $\dot{\gamma}_s(t)$ is mapped onto $\dot{\gamma}_\pi(t)$. Under these maps a Euclidean parallel family of unit vectors along γ_π corresponds to a family $Y(t) \in S_{\gamma_S(t)}$. We must show that this family is parallel in the sense of (1), §5. Let τ denote the angle between $\dot{\gamma}_S(t)$ and $Y(t)$. Then

$$\begin{aligned}\dot{\tau}(t) &= -\text{curvature of } \gamma_\pi \text{ at } \gamma_\pi(t)\\ &= -\text{geodesic curvature of } \gamma_S \text{ at } \gamma_S(t)\\ &= -(\xi \times \dot{\gamma}_S \cdot \ddot{\gamma}_S)(t).\end{aligned}$$



We can choose the coordinates (u, v) near $\gamma_S(t_0)$ such that for t close to t_0

$$u(\gamma_S(t)) = t, \qquad v(\gamma_S(t)) = \text{const.}, \qquad g_{\gamma_S(t)}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = 0.$$

(For example, let $r \to \delta_t(r)$ be a geodesic in S starting at $\gamma_S(t)$ perpendicular to γ_S; small pieces of these geodesics fill up (disjointly) a neigborhood of $\gamma_S(t_0)$; the mapping $\delta_t(r) \to (t, r)$ is a coordinate system with the desired properties.) Writing $Y(t) = Y^1(t)\, r_u + Y^2(t) r_v$ (using notation from previous exercise), we have

$$Y^1(t) = \cos \tau(t), \qquad Y^2(t) = G^{-1/2} \sin \tau(t) \tag{1}$$

and shall now verify (2), §5. By (2), §9 we have

$$2 \sum_l g_{lk} \Gamma_{ij}^l = \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij}.$$

On the curve γ_S we have $E \equiv 1$, $F \equiv 0$, so

$$\Gamma_{11}^1 = 0, \qquad \Gamma_{11}^2 = -\frac{E_v}{2G}, \qquad \Gamma_{12}^1 = \frac{E_v}{2},$$

$$\Gamma_{22}^1 = F_v - \frac{G_u}{2}, \qquad \Gamma_{22}^2 = \frac{G_v}{2G}, \qquad \Gamma_{12}^2 = \frac{G_u}{2G}.$$

Thus we must verify

$$\dot{Y}^1 + \frac{1}{2} E_v Y^2 = 0, \qquad \dot{Y}^2 - \frac{E_v}{2G} Y^1 + \frac{G_u}{2G} Y^2 = 0. \tag{2}$$

But using formulas from Exercise F.2 we find

$$\dot{\tau}(t) = -(\xi \times \dot{\gamma}_S \cdot \ddot{\gamma}_S)(t) = \tfrac{1}{2}(G^{-1/2}E_v)(\gamma_S(t)))$$

and now equations (2) follow directly from (1).

G. The Hyperbolic Plane

1. (i) and (ii) are obvious. (iii) is clear since

$$\frac{x'(t)^2}{(1 - x(t)^2)^2} \leqslant \frac{x'(t)^2 + y'(t)^2}{(1 - x(t)^2 - y(t)^2)^2}$$

where $\gamma(t) = (x(t), y(t))$. For (iv) let $z \in D$, $u \in D_z$, and let $z(t)$ be a curve with $z(0) = z$, $z'(0) = u$. Then

$$d\varphi_z(u) = \left\{\frac{d}{dt}\varphi(z(t))\right\}_{t=0} = \frac{z'(0)}{(\bar{b}z + \bar{a})^2} \qquad \text{at} \quad \varphi \cdot z,$$

and $g(d\varphi(u), d\varphi(u)) = g(u, u)$ now follows by direct computation. Now (v) follows since φ is conformal and maps lines into circles. The first relation in (vi) is immediate; and writing the expression for $d(0, x)$ as a cross ratio of the points $-1, 0, x, 1$, the expression for $d(z_1, z_2)$ follows since φ in (iv) preserves cross ratio. For (vii) let τ be any isometry of D. Then there exists a φ as in (iv) such that $\varphi\tau^{-1}$ leaves the x-axis pointwise fixed. But then $\varphi\tau^{-1}$ is either the identity or the complex conjugation $z \to \bar{z}$. For (viii) we note that if $r = d(0, z)$, then $|z| = \tanh r$; so the formula for g follows from (ii). Part (ix) follows from

$$v = \frac{1 - |z|^2}{|z - i|^2}, \qquad dw = -2\frac{dz}{(z - i)^2}, \qquad d\bar{w} = -2\frac{d\bar{z}}{(\bar{z} + i)^2}.$$