CHAPTER II

A. On the Geometry of Lie Groups

A.1. (i) follows from $\exp \mathrm{Ad}(x)tX = x \exp tXx^{-1} = L(x)\, R(x^{-1}) \exp tX$ for $X \in \mathfrak{g}$, $t \in \boldsymbol{R}$. For (ii) we note $J(x \exp tX) = \exp(-tX)\, x^{-1}$, so $dJ_x(dL(x)_e X) = -dR(x^{-1})_e X$. For (iii) we observe for X_0, $Y_0 \in \mathfrak{g}$

$$\begin{aligned} \Phi(g \exp tX_0, h \exp sY_0) &= g \exp tX_0 h \exp sY_0 \\ &= gh \exp t\, \mathrm{Ad}(h^{-1})\, X_0 \exp sY_0, \end{aligned}$$

(Continued on next page.)

so

$$d\Phi(dL(g)X_0, dL(h)Y_0) = dL(gh)(\mathrm{Ad}(h^{-1})X_0 + Y_0).$$

Putting $X = dL(g)X_0$, $Y = dL(h)Y_0$, the result follows from (i).

A.2. Suppose $\gamma(t_1) = \gamma(t_2)$ so $\gamma(t_2 - t_1) = e$. Let $L > 0$ be the smallest number such that $\gamma(L) = e$. Then $\gamma(t + L) = \gamma(t)\,\gamma(L) = \gamma(t)$. If τ_L denotes the translation $t \to t + L$, we have $\gamma \circ \tau_L = \gamma$, so

$$\dot{\gamma}(0) = d\gamma\left(\frac{d}{dt}\right)_0 = d\gamma\left(\frac{d}{dt}\right)_L = \dot{\gamma}(L).$$

A.3. The curve σ satisfies $\sigma(t + L) = \sigma(t)$, so as in A.2, $\dot{\sigma}(0) = \dot{\sigma}(L)$.

A.4. Let (p_n) be a Cauchy sequence in G/H. Then if d denotes the distance, $d(p_n, p_m) \to 0$ if $m, n \to \infty$. Let $B_\epsilon(o)$ be a relatively compact ball of radius $\epsilon > 0$ around the origin $o = \{H\}$ in G/H. Select N such that $d(p_N, p_m) < \frac{1}{2}\epsilon$ for $m \geqslant N$ and select $g \in G$ such that $g \cdot p_N = o$. Then $(g \cdot p_m)$ is a Cauchy sequence inside the compact ball $B_\epsilon(o)^-$, hence it, together with the original sequence, is convergent.

A.5. For $X \in \mathfrak{g}$ let $\tilde{X}$ denote the corresponding left invariant vector field on G. From Prop. 1.4 we know that (i) is equivalent to $\nabla_{\tilde{Z}}(\tilde{Z}) = 0$ for all $Z \in \mathfrak{g}$. But by (2), §9 in Chapter I this condition reduces to

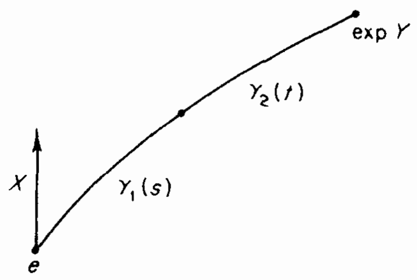
$$g(\tilde{Z}, [\tilde{X}, \tilde{Z}]) = 0 \qquad (X, Z \in \mathfrak{g})$$

which is clearly equivalent to (ii). Next (iii) follows from (ii) by replacing X by $X + Z$. But (iii) is equivalent to $\mathrm{Ad}(G)$-invariance of B so Q is right invariant. Finally, the map $J : x \to x^{-1}$ satisfies $J = R(g^{-1}) \circ J \circ L(g^{-1})$, so $dJ_g = dR(g^{-1})_e \circ dJ_e \circ dL(g^{-1})_g$. Since dJ_e is automatically an isometry, (v) follows.

A.6. Assuming first the existence of ∇, consider the affine transformation $\sigma : g \to \exp \frac{1}{2}Yg^{-1} \exp \frac{1}{2}Y$ of G which fixes the point $\exp \frac{1}{2}Y$ and maps γ_1, the first half of γ, onto the second half, γ_2. Since

$$\sigma = L(\exp \tfrac{1}{2}Y) \circ J \circ L(\exp -\tfrac{1}{2}Y),$$

we have $d\sigma_{\exp \frac{1}{2}Y} = -I$. Let $X^*(t) \in G_{\exp tY}$ $(0 \leqslant t \leqslant 1)$ be the family of vectors parallel with respect to γ such that $X^*(0) = X$. Then σ maps $X^*(s)$ along γ_1 into a parallel field along γ_2 which must be the field $-X^*(t)$ because $d\sigma(X^*(\frac{1}{2})) = -X^*(\frac{1}{2})$. Thus the map $\sigma \circ J = L(\exp \frac{1}{2}Y)\, R(\exp \frac{1}{2}Y)$ sends X into $X^*(1)$, as stated in part (i). Part (ii) now follows from Theorem 7.1, Chapter I, and part (iii) from Prop. 1.4. Now (iv) follows from (ii) and the definition of T and R.



Finally, we prove the existence of ∇. As remarked before Prop. 1.4, the equation $\nabla_{\tilde{X}}(\tilde{Y}) = \frac{1}{2}[\tilde{X}, \tilde{Y}]$ $(X, Y \in \mathfrak{g})$ defines uniquely a left invariant affine connection ∇ on G. Since $\tilde{X}^{R(g)} = (\mathrm{Ad}(g^{-1})X)^{\sim}$, we get

$$\nabla_{\tilde{X}^{R(g)}}(\tilde{Y}^{R(g)}) = \tfrac{1}{2}\{\mathrm{Ad}(g^{-1})[X, Y]\}^{\sim} = (\nabla_{\tilde{X}}(\tilde{Y}))^{R(g)};$$

this we generalize to any vector fields Z, Z' by writing them in terms of $\tilde{X}_i$ $(1 \leqslant i \leqslant n)$. Next

$$\nabla_{J\tilde{X}}(J\tilde{Y}) = J(\nabla_{\tilde{X}}(\tilde{Y})). \qquad (1)$$

Since both sides are right invariant vector fields, it suffices to verify the equation at e. Now $J\tilde{X} = -\bar{X}$ where $\bar{X}$ is right invariant, so the problem is to prove

$$(\nabla_{\bar{X}}(\bar{Y}))_e = -\tfrac{1}{2}[X, Y].$$

For a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ we write $\mathrm{Ad}(g^{-1})Y = \sum_i f_i(g)X_i$. Since $\bar{Y}_g = dR(g)Y = dL(g)\,\mathrm{Ad}(g^{-1})Y$, it follows that $\bar{Y} = \sum_i f_i\tilde{X}_i$, so using ∇_2 and Lemma 4.2 from Chapter I, §4,

$$(\nabla_{\bar{X}}(\bar{Y}))_e = (\nabla_{\tilde{X}}(\bar{Y}))_e = \sum_i (Xf_i)_e\, X_i + \tfrac{1}{2} \sum_i f_i(e)[\tilde{X}, \tilde{X}_i]_e$$

Since $(Xf_i)(e) = \{(d/dt)\, f_i(\exp tX)\}_{t=0}$ and since

$$\left\{\frac{d}{dt}\,\mathrm{Ad}(\exp(-tX))(Y)\right\}_{t=0} = -[X, Y],$$

the expression on the right reduces to $-[X, Y] + \frac{1}{2}[X, Y]$, so (1) follows. As before, (1) generalizes to any vector fields Z, Z'.

The connection ∇ is the 0-connection of Cartan-Schouten [1].

B. The Exponential Mapping

B.1. At the end of §1 it was shown that $GL(2, \boldsymbol{R})$ has Lie algebra $\mathfrak{gl}(2, \boldsymbol{R})$, the Lie algebra of all 2×2 real matrices. Since $\det(e^{tX}) =$

$e^{t\,\mathrm{Tr}(X)}$, Prop. 2.7 shows that $\mathfrak{sl}(2, \boldsymbol{R})$ consists of all 2×2 real matrices of trace 0. Writing

$$X = a\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + b\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = \begin{pmatrix}a & b\\ c & -a\end{pmatrix}$$

a direct computation gives for the Killing form

$$B(X, X) = 8(a^2 + bc) = 4\,\mathrm{Tr}(XX),$$

whence $B(X, Y) = 4\,\mathrm{Tr}(XY)$, and semisimplicity follows quickly. Part (i) is obtained by direct computation. For (ii) we consider the equation

$$e^X = \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} \qquad (\lambda \in \boldsymbol{R}, \quad \lambda \neq 1).$$

Case 1: $\lambda > 0$. Then $\det X < 0$. In fact $\det X = 0$ implies

$$I + X = \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix},$$

so $b = c = 0$, so $a = 0$, contradicting $\lambda \neq 1$. If $\det X > 0$, we deduce quickly from (i) that $b = c = 0$, so $\det X = -a^2$, which is a contradiction. Thus $\det X < 0$ and using (i) again we find the only solution

$$X = \begin{pmatrix}\log\lambda & 0\\ 0 & -\log\lambda\end{pmatrix}.$$

Case 2: $\lambda = -1$. For $\det X > 0$ put $\mu = (\det X)^{1/2}$. Then using (i) the equation amounts to

$$\cos\mu + (\mu^{-1}\sin\mu)a = -1, \qquad (\mu^{-1}\sin\mu)b = 0,$$
$$\cos\mu - (\mu^{-1}\sin\mu)a = -1, \qquad (\mu^{-1}\sin\mu)c = 0.$$

These equations are satisfied for

$$\mu = (2n+1)\pi \quad (n \in \boldsymbol{Z}), \qquad \det X = -a^2 - bc = (2n+1)^2\,\pi^2.$$

This gives infinitely many choices for X as claimed.

Case 3: $\lambda < 0$, $\lambda \neq -1$. If $\det X = 0$, then (i) shows $b = c = 0$, so $a = 0$; impossible. If $\det X > 0$ and we put $\mu = (\det X)^{1/2}$, (i) implies

$$\cos\mu + (\mu^{-1}\sin\mu)a = \lambda, \qquad (\mu^{-1}\sin\mu)b = 0,$$
$$\cos\mu - (\mu^{-1}\sin\mu)a = \lambda^{-1}, \qquad (\mu^{-1}\sin\mu)c = 0.$$

Since $\lambda \neq \lambda^{-1}$, we have $\sin \mu \neq 0$. Thus $b = c = 0$, so $\det X = -a^2$, which is impossible. If $\det X < 0$ and we put $\mu = (-\det X)^{1/2}$, we get from (i) the equations above with sin and cos replaced by sinh and cosh. Again $b = c = 0$, so $\det X = -a^2 = -\mu^2$; thus $a = \pm\mu$, so

$$\cosh \mu \pm \sinh \mu = \lambda, \qquad \cosh \mu \mp \sinh \mu = \lambda^{-1},$$

contradicting $\lambda < 0$. Thus there is no solution in this case, as stated.

B.2. The Killing form on $\mathfrak{sl}(2, \boldsymbol{R})$ provides a bi-invariant pseudo-Riemannian structure with the properties of Exercise A.5. Thus (i) follows from Exercise B.1. Each $g \in \boldsymbol{SL}(2, \boldsymbol{R})$ can be written $g = kp$ where $k \in \boldsymbol{SO}(2)$ and p is positive definite. Clearly $k = \exp T$ where $T \in \mathfrak{sl}(2, \boldsymbol{R})$; and using diagonalization, $p = \exp X$ where $X \in \mathfrak{sl}(2, \boldsymbol{R})$. The formula $g = \exp T \exp X$ proves (ii).

B.3. Follow the hint.

B.4. Considering one-parameter subgroups it is clear that $\mathfrak{g}$ consists of the matrices

$$X(a, b, c) = \begin{pmatrix} 0 & c & 0 & a \\ -c & 0 & 0 & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (a, b, c \in \boldsymbol{R}).$$

Then $[X(a, b, c), X(a_1, b_1, c_1)] = X(cb_1 - c_1 b, c_1 a - ca_1, 0)$, so $\mathfrak{g}$ is readily seen to be solvable. A direct computation gives

$$\exp X(a, b, c) = \begin{pmatrix} \cos c & \sin c & 0 & c^{-1}(a \sin c - b \cos c + b) \\ -\sin c & \cos c & 0 & c^{-1}(b \sin c + a \cos c - a) \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus $\exp X(a, b, 2\pi)$ is the same point in G for all $a, b \in \boldsymbol{R}$, so exp is not injective. Similarly, the points in G with $\gamma = n2\pi$ $(n \in \boldsymbol{Z})$ $\alpha^2 + \beta^2 > 0$ are not in the range of exp. This example occurs in Auslander and MacKenzie [1]; the exponential mapping for a solvable group is systematically investigated in Dixmier [2].

B.5. Let N_0 be a bounded star-shaped open neighborhood of $0 \in \mathfrak{g}$ which exp maps diffeomorphically onto an open neighborhood N_e of e in G. Let $N^* = \exp(\frac{1}{2}N_0)$. Suppose S is a subgroup of G contained in N^*, and let $s \neq e$ in S. Then $s = \exp X$ $(X \in \frac{1}{2}N_0)$. Let $k \in \boldsymbol{Z}^+$ be such that $X, 2X, \ldots, kX \in \frac{1}{2}N_0$ but $(k + 1)X \notin \frac{1}{2}N_0$. Since N_0 is star-shaped, $(k + 1)X \in N_0$; but since $s^{k+1} \in N^*$, we have $s^{k+1} = \exp Y$, $Y \in \frac{1}{2}N_0$. Since exp is one-to-one on N_0, $(k + 1)X = Y \in \frac{1}{2}N_0$, which is a contradiction.

C. Subgroups and Transformation Groups

C.1. The proofs given in Chapter X for $SU^*(2n)$ and $Sp(n, C)$ generalize easily to the other subgroups.

C.2. Let G be a commutative connected Lie group, $(\tilde{G}, \pi)$ its universal covering group. By facts stated during the proof of Theorem 1.11, $\tilde{G}$ is topologically isomorphic to a Euclidean group R^p. Thus G is topologically isomorphic to a factor group of R^p and by a well-known theorem† on topological groups (e.g. Bourbaki [1], Chap. VII) this factor group is topologically isomorphic to $R^n \times T^m$. Thus by Theorem 2.6, G is analytically isomorphic to $R^n \times T^m$.

For the last statement let $\bar{\gamma}$ be the closure of γ in H. By the first statement and Theorem 2.3, $\bar{\gamma} = R^n \times T^m$ for some $n, m \in Z^+$. But γ is dense in $\bar{\gamma}$, so either $n = 1$ and $m = 0$ (γ closed) or $n = 0$ ($\bar{\gamma}$ compact).

C.3. By Theorem 2.6, I is analytic and by Lemma 1.12, dI is injective. Q.E.D.

C.4. The mapping ψ_g turns $g \cdot N_0$ into a manifold which we denote by $(g \cdot N_0)_x$. Similarly, $\psi_{g'}$ turns $g' \cdot N_0$ into a manifold $(g' \cdot N_0)_y$. Thus we have two manifolds $(g \cdot N_0 \cap g' \cdot N_0)_x$ and $(g \cdot N_0 \cap g' \cdot N_0)_y$ and must show that the identity map from one to the other is analytic. Consider the analytic section maps

$$\sigma_g : (g \cdot N_0)_x \to G, \qquad \sigma_{g'} : (g' \cdot N_0)_y \to G$$

defined by

$$\sigma_g(g \exp(x_1X_1 + \dots + x_rX_r) \cdot p_0) = g \exp(x_1X_1 + \dots + x_rX_r),$$
$$\sigma_{g'}(g' \exp(y_1X_1 + \dots + y_rX_r) \cdot p_0) = g' \exp(y_1X_1 + \dots + y_rX_r),$$

and the analytic map

$$J_g : \pi^{-1}(g \cdot N_0) \to (g \cdot N_0)_x \times H$$

given by

$$J_g(z) = (\pi(z), [\sigma_g(\pi(z))]^{-1}z).$$

Furthermore, let $P : (g \cdot N_0)_x \times H \to (g \cdot N_0)_x$ denote the projection on the first component. Then the identity mapping

$$I : (g \cdot N_0 \cap g' \cdot N_0)_y \to (g \cdot N_0 \cap g' \cdot N_0)_x$$

can be factored:

$$(g \cdot N_0 \cap g' \cdot N_0)_y \xrightarrow{\sigma_{g'}} \pi^{-1}(g \cdot N_0) \xrightarrow{J_g} (g \cdot N_0)_x \times H \xrightarrow{P} (g \cdot N_0)_x.$$

† See "Some Details," p. 586.

In fact, if $p \in g \cdot N_0 \cap g' \cdot N_0$, we have

$$p = g \exp(x_1 X_1 + \dots + x_r X_r) \cdot p_0 = g' \exp(y_1 X_1 + \dots + y_r X_r) \cdot p_0,$$

so for some $h \in H$,

$$\begin{aligned} P(J_g(\sigma_{g'}(p))) &= P(J_g(g' \exp(y_1 X_1 + \dots + y_r X_r))) \\ &= P(\pi(g' \exp(y_1 X_1 + \dots + y_r X_r)), h) \\ &= P(\pi(g \exp(x_1 X_1 + \dots + x_r X_r)), h) \\ &= g \exp(x_1 X_1 + \dots + x_r X_r)) \cdot p_0. \end{aligned}$$

Thus I is composed of analytic maps so is analytic, as desired.

C.5. The subgroup $H = G_p$ of G leaving p fixed is closed, so G/H is a manifold. The map $I : G/H \to M$ given by $I(gH) = g \cdot p$ gives a bijection of G/H onto the orbit $G \cdot p$. Carrying the differentiable structure over on $G \cdot p$ by means of I, it remains to prove that $I : G/H \to M$ is everywhere regular. Consider the maps on the diagram

$$\begin{array}{ccc} & G & \\ {\scriptstyle \pi} \swarrow & & \searrow {\scriptstyle \beta} \\ G/H & \xrightarrow[I]{} & M \end{array}$$

where $\pi(g) = gH$, $\beta(g) = g \cdot p$ so $\beta = I \circ \pi$. If we restrict π to a local cross section, we can write $I = \beta \circ \pi^{-1}$ on a neighborhood of the origin in G/H. Thus I is C^∞ near the origin, hence everywhere. Moreover, the map $d\beta_e : \mathfrak{g} \to M_p$ has kernel $\mathfrak{h}$, the Lie algebra of H (cf. proof of Prop. 4.3). Since $d\pi_e$ maps $\mathfrak{g}$ onto $(G/H)_H$ with kernel $\mathfrak{h}$ and since $d\beta_e = dI_H \circ d\pi_e$, wee that dI_H is one-to-one. Finally, if $T(g)$ denotes the diffeomorphism $m \to g \cdot m$ of M, we have $I = T(g) \circ I \circ \tau(g^{-1})$, whence

$$dI_{gH} = dT(g)_p \circ dI_H \circ d\tau(g^{-1})_{gH},$$

so I is everywhere regular.

C.6. By local connectedness each component of G is open. It acquires an analytic structure from that of G_0 by left translation. In order to show the map $\varphi : (x, y) \to xy^{-1}$ analytic at a point $(x_0, y_0) \in G \times G$ let G_1 and G_2 denote the components of G containing x_0 and y_0, respectively. If $\varphi_0 = \varphi \mid G_0 \times G_0$ and $\psi = \varphi \mid G_1 \times G_2$, then

$$\psi = L(x_0 y_0^{-1}) \circ I(y_0) \circ \varphi_0 \circ L(x_0^{-1}, y_0^{-1}),$$

where $I(y_0)(x) = y_0 x y_0^{-1}$ $(x \in G_0)$. Now $I(y_0)$ is a continuous automorphism of the Lie group G_0, hence by Theorem 2.6, analytic; so the expression for ψ shows that it is analytic.

C.8. If N with the indicated properties exists we may, by translation, assume it passes through the origin $o = \{H\}$ in M. Let L be the subgroup $\{g \in G : g \cdot N = N\}$. If $g \in G$ maps o into N, then $gN \cap N \neq \emptyset$; so by assumption, $gN = N$. Thus $L = \pi^{-1}(N)$ where $\pi : G \to G/H$ is the natural map. Using Theorem 15.5, Chapter I we see that L can be given the structure of a submanifold of G with a countable basis and by the transitivity of G on M, $L \cdot o = N$. By C.7, L has the desired property. For the converse, define $N = L \cdot o$ and use Prop. 4.4 or Exercise C.5. Clearly, if $gN \cap N \neq \emptyset$, then $g \in L$, so $gN = N$.

For more information on the primitivity notion which goes back to Lie see e.g. Golubitsky [1].

D. Closed Subgroups

D.1. $\boldsymbol{R}^2/\Gamma$ is a torus (Exercise C.2), so it suffices to take a line through 0 in $\boldsymbol{R}^2$ whose image in the torus is dense.

D.2. $\mathfrak{g}$ has an $\mathrm{Int}(\mathfrak{g})$-invariant positive definite quadratic form Q. The proof of Prop. 6.6 now shows $\mathfrak{g} = \mathfrak{z} + \mathfrak{g}'$ ($\mathfrak{z}$ = center of $\mathfrak{g}$, $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ compact and semisimple). The groups $\mathrm{Int}(\mathfrak{g})$ and $\mathrm{Int}(\mathfrak{g}')$ are analytic subgroups of $\boldsymbol{GL}(\mathfrak{g})$ with the same Lie algebra so coincide.

D.3. We have

$$\alpha_{0,\frac{1}{3}}(c_1, c_2, s) = (c_1, e^{2\pi i/3}c_2, s)$$

$$\begin{aligned}(a_1, a_2, r)&(c_1, c_2, s)(a_1, a_2, r)^{-1}\\ &= (a_1(1 - e^{2\pi i s}) + c_1 e^{2\pi i r}, a_2(1 - e^{2\pi i h s}) + c_2 e^{2\pi i h r}, s)\end{aligned}$$

so $\alpha_{0,\frac{1}{3}}$ is not an inner automorphism, and $A_{0,\frac{1}{3}} \notin \mathrm{Int}(\mathfrak{g})$. Now let $s_n \to 0$ and let $t_n = hs_n + hn$. Select a sequence $(n_k) \subset \boldsymbol{Z}$ such that $hn_k \to \frac{1}{3}$ (mod 1) (Kronecker's theorem), and let τ_k be the unique point in $[0, 1)$ such that $t_{n_k} - \tau_k \in \boldsymbol{Z}$. Putting $s_k = s_{n_k}$, $t_k = t_{n_k}$, we have

$$\alpha_{s_k, t_k} = \alpha_{s_k, \tau_k} \to \alpha_{0,\frac{1}{3}}.$$

Note: G is a subgroup of $H \times H$ where $H = \begin{pmatrix} 1 & 0 \\ c & \alpha \end{pmatrix}$, $c \in \boldsymbol{C}$, $|\alpha| = 1$.

E. Invariant Differential Forms

E.1. The affine connection on G given by $\nabla_{\tilde{X}}(\tilde{Y}) = \frac{1}{2}[\tilde{X}, \tilde{Y}]$ is torsion free; and by (5), §7, Chapter I, if ω is a left invariant 1-form,

$$\nabla_{\tilde{X}}(\omega)(\tilde{Y}) = -\omega(\nabla_{\tilde{X}}(\tilde{Y})) = -\tfrac{1}{2}\omega(\theta(\tilde{X})(\tilde{Y})) = \tfrac{1}{2}(\theta(\tilde{X})\omega)(\tilde{Y}),$$

so $\nabla_{\tilde{X}}(\omega) = \frac{1}{2}\theta(\tilde{X})(\omega)$ for all left invariant forms ω. Now use Exercise C.4 in Chapter I.

E.2. The first relation is proved as (4), §7. For the other we have $g{}^tg = I$, so $(dg){}^tg + g{}^t(dg) = 0$. Hence $(g^{-1}\,dg) + {}^t(dg)({}^tg)^{-1} = 0$ and $\Omega + {}^t\Omega = 0$.

For $U(n)$ we find similarly for $\Omega = g^{-1}\,dg$,

$$d\Omega + \Omega \wedge \Omega = 0, \qquad \Omega + {}^t\bar{\Omega} = 0.$$

For $Sp(n) \subset U(2n)$ we recall that $g \in Sp(n)$ if and only if

$$g{}^t\bar{g} = I_{2n}, \qquad gJ_n{}^tg = J_n$$

(cf. Chapter X). Then the form $\Omega = g^{-1}\,dg$ satisfies

$$d\Omega + \Omega \wedge \Omega = 0, \qquad \Omega + {}^t\bar{\Omega} = 0, \qquad \Omega J_n + J_n{}^t\Omega = 0.$$

E.3. A direct computation gives

$$g^{-1}\,dg = \begin{pmatrix} 0 & dx & dz - x\,dy \\ 0 & 0 & dy \\ 0 & 0 & 0 \end{pmatrix}$$

and the result follows.