A. Manifolds

2. Let M be a connected manifold and p, q two points in M. Then there exists a diffeomorphism Φ of M onto itself such that $\Phi(p) = q$.

3. Let M be a Hausdorff space and let δ and δ' be two differentiable structures on M. Let $\mathfrak{F}$ and $\mathfrak{F}'$ denote the corresponding sets of C^∞ functions. Then $\delta = \delta'$ if and only if $\mathfrak{F} = \mathfrak{F}'$.

Deduce that the real line $\boldsymbol{R}$ with its ordinary topology has infinitely many different differentiable structures.

4. Let Φ be a differentiable mapping of a manifold M onto a manifold N. A vector field X on M is called *projectable* (Koszul [1]) if there exists a vector field Y on N such that $d\Phi \cdot X = Y$.

(i) Show that X is projectable if and only if $X\mathfrak{F}_0 \subset \mathfrak{F}_0$ where $\mathfrak{F}_0 = \{f \circ \Phi : f \in C^\infty(N)\}$.

(ii) A necessary condition for X to be projectable is that

$$d\Phi_p(X_p) = d\Phi_q(X_q) \tag{1}$$

whenever $\Phi(p) = \Phi(q)$. If, in addition, $d\Phi_p(M_p) = N_{\Phi(p)}$ for each $p \in M$, this condition is also sufficient.

(iii) Let $M = \boldsymbol{R}$ with the usual differentiable structure and let N be the topological space $\boldsymbol{R}$ with the differentiable structure obtained by requiring the homeomorphism $\psi : x \to x^{1/3}$ of M onto N to be a diffeomorphism. In this case the identity mapping $\Phi : x \to x$ is a differentiable mapping of M onto N. The vector field $X = \partial/\partial x$ on M is not projectable although (1) is satisfied.

5. Deduce from §3.1 that diffeomorphic manifolds have the same dimension.

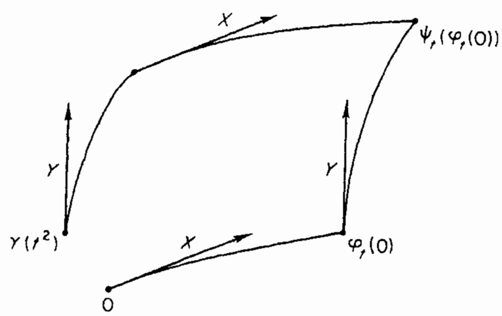
7. Let M be a manifold, $p \in M$, and X a vector field on M such that $X_p \neq 0$. Then there exists a local chart $\{x_1, ..., x_m\}$ on a neighborhood U of p such that $X = \partial/\partial x_1$ on U. Deduce that the differential equation $Xu = f\ (f \in C^\infty(M))$ has a solution u in a neighborhood of p.

8. Let M be a manifold and X, Y two vector fields both $\neq 0$ at a point $o \in M$. For p close to o and s, $t \in \boldsymbol{R}$ sufficiently small let $\varphi_s(p)$ and $\psi_t(p)$ denote the integral curves through p of X and Y, respectively. Let

$$\gamma(t) = \psi_{-\sqrt{t}}(\varphi_{-\sqrt{t}}(\psi_{\sqrt{t}}(\varphi_{\sqrt{t}}(o)))).$$

Prove that

$$[X, Y]_o = \lim_{t \to 0} \dot{\gamma}(t)$$



(Hint: The curves $t \to \varphi_t(\varphi_s(p))$ and $t \to \varphi_{t+s}(p)$ must coincide; deduce $(X^n f)(p) = [d^n/dt^n f(\varphi_t \cdot p)]_{t=0}$).

B. The Lie Derivative and the Interior Product

1. Let M be a manifold, X a vector field on M. The Lie derivative $\theta(X) : Y \to [X, Y]$ which maps $\mathfrak{D}^1(M)$ into itself can be extended uniquely to a mapping of $\mathfrak{D}(M)$ into itself such that:

(i) $\theta(X)f = Xf$ for $f \in C^\infty(M)$.

(ii) $\theta(X)$ is a derivation of $\mathfrak{D}(M)$ preserving type of tensors.

(iii) $\theta(X)$ commutes with contractions.

2. Let Φ be a diffeomorphism of a manifold M onto itself. Then Φ induces a unique type-preserving automorphism $T \to \Phi \cdot T$ of the tensor algebra $\mathfrak{D}(M)$ such that:

(i) The automorphism commutes with contractions.

(ii) $\Phi \cdot X = X^\Phi$, $(X \in \mathfrak{D}^1(M))$, $\Phi \cdot f = f^\Phi$, $(f \in C^\infty(M))$.

Prove that $\Phi \cdot \omega = (\Phi^{-1})^* \omega$ for $\omega \in \mathfrak{D}_*(M)$.

3. Let g_t be a one-parameter Lie transformation group of M and denote by X the vector field on M induced by g_t (Chapter II, §3). Then

$$\theta(X)T = \lim_{t \to 0} \frac{1}{t}(T - g_t \cdot T)$$

for each tensor field T on M ($g_t \cdot T$ is defined in Exercise 2).

4. The Lie derivative $\theta(X)$ on a manifold M has the following properties:

(i) $\theta([X, Y]) = \theta(X)\,\theta(Y) - \theta(Y)\,\theta(X)$, $\quad X, Y \in \mathfrak{D}^1(M)$.

(ii) $\theta(X)$ commutes with the alternation $A : \mathfrak{D}_*(M) \to \mathfrak{A}(M)$ and therefore induces a derivation of the Grassmann algebra of M.

(iii) $\theta(X)\,d = d\theta(X)$, that is, $\theta(X)$ commutes with exterior differentiation.

5. For $X \in \mathfrak{D}^1(M)$ there is a unique linear mapping $i(X) : \mathfrak{A}(M) \to \mathfrak{A}(M)$, the *interior product*, satisfying:

(i) $i(X)f = 0$ for $f \in C^\infty(M)$.

(ii) $i(X)\omega = \omega(X)$ for $\omega \in \mathfrak{A}_1(M)$.

(iii) $i(X) : \mathfrak{A}_r(M) \to \mathfrak{A}_{r-1}(M)$ and

$$i(X)(\omega_1 \wedge \omega_2) = i(X)(\omega_1) \wedge \omega_2 + (-1)^r \omega_1 \wedge i(X)(\omega_2)$$

if $\omega_1 \in \mathfrak{A}_r(M)$, $\omega_2 \in \mathfrak{A}(M)$.

6. (cf. H. Cartan [1]). Prove that if $X, Y \in \mathfrak{D}^1(M)$, $\omega_1, \ldots, \omega_r \in \mathfrak{A}_1(M)$,

(i) $i(X)^2 = 0$.

(ii) $i(X)(\omega_1 \wedge \ldots \wedge \omega_r) = \sum_{1 \leq k \leq r} (-1)^{k+1} \omega_k(X)\, \omega_1 \wedge \ldots \wedge \hat{\omega}_k \wedge \ldots \wedge \omega_r;$

$\omega_i \in \mathfrak{A}_1(M)$.

(iii) $i([X, Y]) = \theta(X)\, i(Y) - i(Y)\, \theta(X)$.

(iv) $\theta(X) = i(X)\, d + d\, i(X)$.

C. Affine Connections

2. Let ∇ be the affine connection on R^n determined by $\nabla_X(Y) = 0$ for $X = \partial/\partial x_i$, $Y = \partial/\partial x_j$, $1 \leq i, j \leq n$. Find the corresponding affine transformations.

4. Let M be a manifold with a torsion-free affine connection ∇. Suppose $X_1, \ldots, X_m$ is a basis for the vector fields on an open subset U of M. Let the forms $\omega^1, \ldots, \omega^m$ on U be determined by $\omega^i(X_j) = \delta^i{}_j$. Prove the formula

$$d\theta = \sum_{i=1}^{m} \omega^i \wedge \nabla_{X_i}(\theta)$$

for each differential form θ on U.

5. Let S be a surface in R^3, X and Y two vector fields on S. Let $s \in S$, $X_s \neq 0$ and $t \to \gamma(t)$ a curve on S through s such that $\dot{\gamma}(t) = X_{\gamma(t)}$, $\gamma(0) = s$. Viewing $Y_{\gamma(t)}$ as a vector in R^3 and letting $\pi_s : R^3 \to S_s$ denote the orthogonal projection put

$$\nabla'_X(Y)_s = \pi_s\left(\lim_{t \to 0} \frac{1}{t}\,(Y_{\gamma(t)} - Y_s)\right).$$

Prove that this defines an affine connection on S.

D. Submanifolds

1. Let M and N be differentiable manifolds and Φ a differentiable mapping of M into N. Consider the mapping $\varphi : m \to (m, \Phi(m))$ $(m \in M)$ and the graph

$$G_\Phi = \{(m, \Phi(m)) : m \in M\}$$

of Φ with the topology induced by the product space $M \times N$. Then φ is a homeomorphism of M onto G_Φ and if the differentiable structure of M is transferred to G_Φ by φ, the graph G_Φ becomes a closed submanifold of $M \times N$.

2. Let N be a manifold and M a topological space, $M \subset N$ (as sets).

Show that there exists at most one differentiable structure on the topological space M such that M is a submanifold of N.

3. Using the figure 8 as a subset of R^2 show that

(i) A closed connected submanifold of a connected manifold does not necessarily carry the relative topology.

(ii) A subset M of a connected manifold N may have two different topologies and differentiable structures such that in both cases M is a submanifold of N.

4. Let M be a submanifold of a manifold N and suppose $M = N$ (as sets). Assuming M to have a countable basis for the open sets, prove that $M = N$ (as manifolds). (Use Prop. 3.2 and Lemma 3.1, Chapter II.)

E. The Hyperbolic Plane

1. Let D be the open disk $|z| < 1$ in R^2 with the usual differentiable structure but given the Riemannian structure

$$g(u, v) = \frac{(u, v)}{(1 - |z|^2)^2} \qquad (u, v \in D_z)$$

$(\, , \,)$ denoting the usual inner product on R^2.

(i) Show that the angle between u and v in the Riemannian structure g coincides with the Euclidean angle.

(ii) Show that the Riemannian structure can be written

$$g = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2} \qquad (z = x + iy).$$

(iii) Show that the arc length L satisfies

$$L(\gamma_0) \leqslant L(\gamma)$$

if γ is any curve joining the origin 0 and x $(0 < x < 1)$ and $\gamma_0(t) = tx$ $(0 \leqslant t \leqslant 1)$.

(iv) Show that the transformation

$$\varphi : z \to \frac{az + b}{\bar{b}z + \bar{a}} \qquad (|a|^2 - |b|^2 = 1)$$

is an isometry of D.

(v) Deduce from (iii) and (iv) that the geodesics in D are the circular arcs perpendicular to the boundary $|z| = 1$.

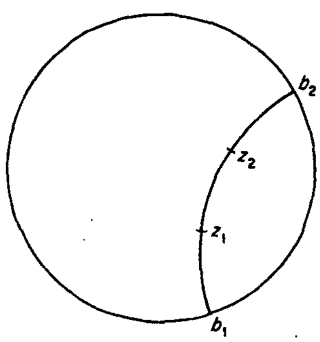
(vi) Prove from (iii) that

$$d(0, z) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|} \qquad (z \in D)$$

and using (iv) that

$$d(z_1, z_2) = \frac{1}{2} \log \left(\frac{z_1 - b_2}{z_1 - b_1} : \frac{z_2 - b_2}{z_2 - b_1} \right) \qquad (z_1, z_2 \in D)$$

with b_1 and b_2 as in the figure.



(vii) Show that the maps φ in (iv) together with the complex conjugation $z \to \bar{z}$ generate the group of all isometries of D.