SOLUTIONS TO EXERCISES

CHAPTER I

A. Manifolds

A.2. If p_1, $p_2 \in M$ are sufficiently close within a coordinate neighborhood U, there exists a diffeomorphism mapping p_1 to p_2 and leaving $M - U$ pointwise fixed. Now consider a curve segment $\gamma(t)$ $(0 \leqslant t \leqslant 1)$ in M joining p to q. Let t^* be the supremum of those t for which there exists a diffeomorphism of M mapping p on $\gamma(t)$. The initial remark shows first that $t^* > 0$, next that $t^* = 1$, and finally that t^* is reached as a maximum.

A.3. The "only if" is obvious and "if" follows from the uniqueness in Prop. 1.1. Now let $\mathfrak{F} = C^\infty(R)$ where R is given the ordinary differentiable structure. If n is an odd integer, let $\mathfrak{F}^n$ denote the set of functions $x \to f(x^n)$ on R, $f \in \mathfrak{F}$ being arbitrary. Then $\mathfrak{F}^n$ satisfies $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_3$. Since $\mathfrak{F}^n \neq \mathfrak{F}^m$ for $n \neq m$, the corresponding δ^n are all different.

A.4. (i) If $d\Phi \cdot X = Y$ and $f \in C^\infty(N)$, then $X(f \circ \Phi) = (Yf) \circ \Phi \in \mathfrak{F}_0$. On the other hand, suppose $X\mathfrak{F}_0 \subset \mathfrak{F}_0$. If $F \in \mathfrak{F}_0$, then $F = g \circ \Phi$ where $g \in C^\infty(N)$ is unique. If $f \in C^\infty(N)$, then $X(f \circ \Phi) = g \circ \Phi$ ($g \in C^\infty(N)$ unique), and $f \to g$ is a derivation, giving Y.

(ii) If $d\Phi \cdot X = Y$, then $Y_{\Phi(p)} = d\Phi_p(X_p)$, so necessity follows. Suppose $d\Phi_p(M_p) = N_{\Phi(p)}$ for each $p \in M$. Define for $r \in N$, $Y_r = d\Phi_p(X_p)$ if $r = \Phi(p)$. In order to show that $Y : r \to Y_r$ is differentiable we use coordinates around p and around $r = \Phi(p)$ such that Φ has the expression $(x_1, ..., x_m) \to (x_1, ..., x_n)$. Writing

$$X = \sum_1^m a_i(x_1, ..., x_m) \frac{\partial}{\partial x_i},$$

we have for q sufficiently near p

$$d\Phi_q(X_q) = \sum_1^n a_i(x_1(q), \ldots, x_m(q)) \left(\frac{\partial}{\partial x_i}\right)_{\Phi(q)},$$

so condition (1) implies that for $1 \leqslant i \leqslant n$, a_i is constant in the last $m - n$ arguments. Hence

$$Y = \sum_1^n a_i(x_1, \ldots, x_n, x_{n+1}(p), \ldots, x_m(p)) \frac{\partial}{\partial x_i}.$$

(iii) $f \in C^\infty(N)$ if and only if $f \circ \psi \in C^\infty(R)$. If $f(x) = x^3$, then $f \circ \psi(x) = x$, $(f' \circ \psi)(x) = 3x^{2/3}$, so $f \in C^\infty(N)$, $f' \notin C^\infty(N)$. Hence $f \circ \Phi \in \mathfrak{F}_0$, but $X(f \circ \Phi) \notin \mathfrak{F}_0$; so by (i), X is not projectable.

A.5. Obvious.

A.7. We can assume $M = R^m$, $p = 0$, and that $X_0 = (\partial/\partial t_1)_0$ in terms of the standard coordinate system $\{t_1, \ldots, t_m\}$ on R^m. Consider the integral curve $\varphi_t(0, c_2, \ldots, c_m)$ of X through $(0, c_2, \ldots, c_m)$. Then the mapping $\psi : (c_1, \ldots, c_m) \to \varphi_{c_1}(0, c_2, \ldots, c_m)$ is C^∞ for small c_i, $\psi(0, c_2, \ldots, c_m) = (0, c_2, \ldots, c_m)$, so

$$d\psi_0 \left(\frac{\partial}{\partial c_i}\right) = \left(\frac{\partial}{\partial t_i}\right)_0 \qquad (i > 1).$$

Also

$$d\psi_0 \left(\frac{\partial}{\partial c_1}\right)_0 = \left(\frac{\partial \varphi_{c_1}}{\partial c_1}\right)(0) = X_0 = \left(\frac{\partial}{\partial t_1}\right)_0.$$

Thus ψ can be inverted near 0, so $\{c_1, \ldots, c_m\}$ is a local coordinate system. Finally, if $c = (c_1, \ldots, c_m)$,

$$\begin{aligned}
\left(\frac{\partial}{\partial c_1}\right)_{\psi(c)} f &= \left(\frac{\partial (f \circ \psi)}{\partial c_1}\right)_c \\
&= \lim_{h \to 0} \frac{1}{h} [f(\varphi_{c_1+h}(0, c_2, \ldots, c_m)) - f(\varphi_{c_1}(0, c_2, \ldots, c_m))] \\
&= (Xf)(\psi(c))
\end{aligned}$$

so $X = \partial/\partial c_1$.

A.8. Let $f \in C^\infty(M)$. Writing $\sim$ below when in an equality we omit terms of higher order in s or t, we have

$$\begin{aligned} & f(\psi_{-t}(\varphi_{-s}(\psi_t(\varphi_s(o))))) - f(o) \\ &\quad = f(\psi_{-t}(\varphi_{-s}(\psi_t(\varphi_s(o))))) - f(\varphi_{-s}(\psi_t(\varphi_s(o)))) \\ &\qquad + f(\varphi_{-s}(\psi_t(\varphi_s(o)))) - f(\psi_t(\varphi_s(o))) \\ &\qquad + f(\psi_t(\varphi_s(o))) - f(\varphi_s(o)) + f(\varphi_s(o)) - f(o) \\ &\quad \sim -t(Yf)(\varphi_{-s}(\psi_t(\varphi_s(o)))) + \tfrac{1}{2}t^2(Y^2f)(\varphi_{-s}(\psi_t(\varphi_s(o)))) \\ &\qquad - s(Xf)(\psi_t(\varphi_s(o))) + \tfrac{1}{2}s^2(X^2f)(\psi_t(\varphi_s(o))) \\ &\qquad + t(Yf)(\psi_t(\varphi_s(o))) - \tfrac{1}{2}t^2(Y^2f)(\psi_t(\varphi_s(o))) \\ &\qquad + s(Xf)(\varphi_s(o)) - \tfrac{1}{2}s^2(X^2f)(\varphi_s(o)) \\ &\quad \sim st(XYf)(\psi_t(\varphi_s(o))) - st(YXf)(\psi_t(\varphi_s(o))). \end{aligned}$$

This last expression is obtained by pairing off the 1st and 5th term, the 3rd and 7th, the 2nd and 6th, and the 4th and 8th. Hence

$$f(\gamma(t^2)) - f(o) = t^2([X, Y]f)(o) + O(t^3).$$

A similar proof is given in Faber [1].

B. The Lie Derivative and the Interior Product

B.1. If the desired extension of $\theta(X)$ exists and if $C : \mathfrak{D}_1^1(M) \to C^\infty(M)$ is the contraction, then (i), (ii), (iii) imply

$$(\theta(X)\omega)(Y) = X(\omega(Y)) - \omega([X, Y]), \qquad X, Y \in \mathfrak{D}^1(M).$$

Thus we define $\theta(X)$ on $\mathfrak{D}_1(M)$ by this relation and note that $(\theta(X)\omega)(fY) = f(\theta(X)(\omega))(Y)$ $(f \in C^\infty(M))$, so $\theta(X)$ $\mathfrak{D}_1(M) \subset \mathfrak{D}_1(M)$. If U is a coordinate neighborhood with coordinates $\{x_1, \ldots, x_m\}$, $\theta(X)$ induces an endomorphism of $C^\infty(U)$, $\mathfrak{D}^1(U)$, and $\mathfrak{D}_1(U)$. Putting $X_i = \partial/\partial x_i$, $\omega_j = dx_j$, each $T \in \mathfrak{D}_s^r(U)$ can be written

$$T = \sum T_{(i),(j)} X_{i_1} \otimes \ldots \otimes X_{i_r} \otimes \omega_{j_1} \otimes \ldots \otimes \omega_{j_s}$$

with unique coefficients $T_{(i),(j)} \in C^\infty(U)$. Now $\theta(X)$ is uniquely extended to $\mathfrak{D}(U)$ satisfying (i) and (ii). Property (iii) is then verified by induction on r and s. Finally, $\theta(X)$ is defined on $\mathfrak{D}(M)$ by the condition $\theta(X)T \mid U = \theta(X)(T \mid U)$ (vertical bar denoting restriction) because as in the proof of Theorem 2.5 this condition is forced by the requirement that $\theta(X)$ should be a derivation.

B.2. The first part being obvious, we just verify $\Phi \cdot \omega = (\Phi^{-1})^*\omega$. We may assume $\omega \in \mathfrak{D}_1(M)$. If $X \in \mathfrak{D}^1(M)$ and C is the contraction $X \otimes \omega \to \omega(X)$, then $\Phi \circ C = C \circ \Phi$ implies $(\Phi \cdot \omega)(X) = \Phi(\omega(X^{\Phi^{-1}})) = ((\Phi^{-1})^*\omega)(X)$.

B.3. The formula is obvious if $T = f \in C^\infty(M)$. Next let $T = Y \in \mathfrak{D}^1(M)$. If $f \in C^\infty(M)$ and $q \in M$, we put $F(t, q) = f(g_t \cdot q)$ and have

$$F(t, q) - F(0, q) = t \int_0^1 \left(\frac{\partial F}{\partial t}\right)(st, q)\, ds = t\, h(t, q),$$

where $h \in C^\infty(R \times M)$ and $h(0, q) = (Xf)(q)$. Then

$$(g_t \cdot Y)_p f = (Y(f \circ g_t))(g_t^{-1} \cdot p) = (Yf)(g_t^{-1} \cdot p) + t(Yh)(t, g_t^{-1} \cdot p)$$

so

$$\lim_{t \to 0} \frac{1}{t}(Y - g_t \cdot Y)_p f = (XYf)(p) - (YXf)(p),$$

so the formula holds for $T \in \mathfrak{D}^1(M)$. But the endomorphism $T \to \lim_{t \to 0} t^{-1}(T - g_t \cdot T)$ has properties (i), (ii), and (iii) of Exercise B.1; it coincides with $\theta(X)$ on $C^\infty(M)$ and on $\mathfrak{D}^1(M)$, hence on all of $\mathfrak{D}(M)$ by the uniqueness in Exercise B.1.

B.4. For (i) we note that both sides are derivations of $\mathfrak{D}(M)$ commuting with contractions, preserving type, and having the same effect on $\mathfrak{D}^1(M)$ and on $C^\infty(M)$. The argument of Exercise B.1 shows that they coincide on $\mathfrak{D}(M)$.

(ii) If $\omega \in \mathfrak{D}_r(M)$, $Y_1, ..., Y_r \in \mathfrak{D}^1(M)$, then by B.1,

$$(\theta(X)\omega)(Y_1, ..., Y_r) = X(\omega(Y_1, ..., Y_r)) - \sum_i \omega(Y_1, ..., [X, Y_i], ..., Y_r)$$

so $\theta(X)$ commutes with A.

(iii) Since $\theta(X)$ is a derivation of $\mathfrak{A}(M)$ and d is a *skew-derivation* (that is, satisfies (iv) in Theorem 2.5), the commutator $\theta(X)d - d\theta(X)$ is also a skew-derivation. Since it vanishes on f and df ($f \in C^\infty(M)$), it vanishes identically (cf. Exercise B.1). For B.1–B.4, cf. Palais [3].

B.5. This is done by the same method as in Exercise B.1.

B.6. For (i) we note that by (iii) in Exercise B.5, $i(X)^2$ is a derivation. Since it vanishes on $C^\infty(M)$ and $\mathfrak{D}_1(M)$, it vanishes identically; (ii) follows by induction; (iii) follows since both sides are skew-derivations which coincide on $C^\infty(M)$ and on $\mathfrak{A}_1(M)$; (iv) follows because both sides are derivations which coincide on $C^\infty(M)$ and on $\mathfrak{A}_1(M)$.

C.2. If Φ is an affine transformation and we write $d\Phi(\partial/\partial x_j) = \sum_i a_{ij}\, \partial/\partial x_i$, then conditions ∇_1 and ∇_2 imply that each a_{ij} is a constant. If A is the linear transformation (a_{ij}), then $\Phi \circ A^{-1}$ has differential I, hence is a translation B, so $\Phi(X) = AX + B$. The converse is obvious.

C.4. A direct verification shows that the mapping $\delta : \theta \to \sum_1^m \omega_i \wedge \nabla_{X_i}(\theta)$ is a skew-derivation of $\mathfrak{A}(M)$ and that it coincides with d on $C^\infty(M)$. Next let $\theta \in \mathfrak{A}_1(M)$, $X, Y \in \mathfrak{D}^1(M)$. Then, using (5), §7,

$$\begin{aligned} 2\,\delta\theta(X, Y) &= 2 \sum_i (\omega_i \wedge \nabla_{X_i}(\theta))(X, Y) \\ &= \sum_i \omega_i(X)\, \nabla_{X_i}(\theta)(Y) - \omega_i(Y)\, \nabla_{X_i}(\theta)(X) \\ &= \nabla_X(\theta)(Y) - \nabla_Y(\theta)(X) \\ &= X \cdot \theta(Y) - \theta(\nabla_X(Y)) - Y \cdot \theta(X) + \theta(\nabla_Y(X)), \end{aligned}$$

which since the torsion is 0 equals

$$X\theta(Y) - Y \cdot \theta(X) - \theta([X, Y]) = 2\, d\theta(X, Y).$$

Thus $\delta = d$ on $\mathfrak{A}_1(M)$, hence by the above on all of $\mathfrak{A}(M)$.

C.5. Let Z be a vector field on S and $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ vector fields on a neighborhood of s in R^3 extending X, Y, and Z, respectively. The inner product $\langle\, ,\, \rangle$ on R^3 induces a Riemannian structure g on S. If $\tilde{\nabla}$ and ∇ denote the corresponding affine connections on R^3 and S, respectively, we deduce from (2), §9

$$\langle \tilde{Z}_s, \tilde{\nabla}_{\tilde{X}}(\tilde{Y})_s \rangle = g(Z_s, \nabla_X(Y)_s).$$

But

$$\tilde{\nabla}_{\tilde{X}}(\tilde{Y})_s = \lim_{t \to 0} \frac{1}{t} (Y_{\gamma(t)} - Y_s),$$

so we obtain $\nabla = \nabla'$; in particular ∇' is an affine connection on S.

D. Submanifolds

D.1. Let $I : G_\Phi \to M \times N$ denote the identity mapping and $\pi : M \times N \to M$ the projection onto the first factor. Let $m \in M$ and $Z \in (G_\Phi)_{(m, \Phi(m))}$ such that $dI_m(Z) = 0$. Then $Z = (d\varphi)_m(X)$ where $X \in M_m$. Thus $d\pi \circ dI \circ d\varphi(X) = 0$. But since $\pi \circ I \circ \varphi$ is the identity mapping, this implies $X = 0$, so $Z = 0$ and I is regular.

D.2. Immediate from Lemma 3.4.

D.3. Consider the figure 8 given by the formula

$$\gamma(t) = (\sin 2t, \sin t) \qquad (0 \leqslant t \leqslant 2\pi).$$

Let $f(s)$ be an increasing function on R such that

$$\lim_{s \to -\infty} f(s) = 0, \qquad f(0) = \pi, \qquad \lim_{s \to +\infty} f(s) = 2\pi.$$

Then the map $s \to \gamma(f(s))$ is a bijection of R onto the figure 8. Carrying the manifold structure of R over, we get a submanifold of R^2 which is closed, yet does not carry the induced topology. Replacing γ by δ given by $\delta(t) = (-\sin 2t, \text{sint } t)$, we get another manifold structure on the figure.

D.4. Suppose $\dim M < \dim N$. Using the notation of Prop. 3.2, let W be a compact neighborhood of p in M and $W \subset U$. By the countability assumption, countably many such W cover M. Thus by Lemma 3.1, Chapter II, for N, some such W contains an open set in N; contradiction.

G. The Hyperbolic Plane

1. (i) and (ii) are obvious. (iii) is clear since

$$\frac{x'(t)^2}{(1 - x(t)^2)^2} \leqslant \frac{x'(t)^2 + y'(t)^2}{(1 - x(t)^2 - y(t)^2)^2}$$

where $\gamma(t) = (x(t), y(t))$. For (iv) let $z \in D$, $u \in D_z$, and let $z(t)$ be a curve with $z(0) = z$, $z'(0) = u$. Then

$$d\varphi_z(u) = \left\{\frac{d}{dt}\, \varphi(z(t))\right\}_{t=0} = \frac{z'(0)}{(\bar{b}z + \bar{a})^2} \qquad \text{at} \quad \varphi \cdot z,$$

and $g(d\varphi(u), d\varphi(u)) = g(u, u)$ now follows by direct computation. Now (v) follows since φ is conformal and maps lines into circles. The first relation in (vi) is immediate; and writing the expression for $d(0, x)$ as a cross ratio of the points $-1, 0, x, 1$, the expression for $d(z_1, z_2)$ follows since φ in (iv) preserves cross ratio. For (vii) let τ be any isometry of D. Then there exists a φ as in (iv) such that $\varphi\tau^{-1}$ leaves the x-axis pointwise fixed. But then $\varphi\tau^{-1}$ is either the identity or the complex conjugation $z \to \bar{z}$.

CHAPTER II

A. On the Geometry of Lie Groups

A.1. (i) follows from $\exp \operatorname{Ad}(x)tX = x \exp tXx^{-1} = L(x)\, R(x^{-1}) \exp tX$ for $X \in \mathfrak{g}$, $t \in R$. For (ii) we note $J(x \exp tX) = \exp(-tX)\, x^{-1}$, so $dJ_x(dL(x)_e X) = -dR(x^{-1})_e X$. For (iii) we observe for $X_0, Y_0 \in \mathfrak{g}$

$$\Phi(g \exp tX_0, h \exp sY_0) = g \exp tX_0 h \exp sY_0$$
$$= gh \exp t \operatorname{Ad}(h^{-1})\, X_0 \exp sY_0,$$

so

$$d\Phi(dL(g)X_0, dL(h)Y_0) = dL(gh)(\operatorname{Ad}(h^{-1})X_0 + Y_0).$$

Putting $X = dL(g)X_0$, $Y = dL(h)Y_0$, the result follows from (i).

A.2. Suppose $\gamma(t_1) = \gamma(t_2)$ so $\gamma(t_2 - t_1) = e$. Let $L > 0$ be the smallest number such that $\gamma(L) = e$. Then $\gamma(t + L) = \gamma(t)\, \gamma(L) = \gamma(t)$. If τ_L denotes the translation $t \to t + L$, we have $\gamma \circ \tau_L = \gamma$, so

$$\dot\gamma(0) = d\gamma\left(\frac{d}{dt}\right)_0 = d\gamma\left(\frac{d}{dt}\right)_L = \dot\gamma(L).$$

A.3. The curve σ satisfies $\sigma(t + L) = \sigma(t)$, so as in A.2, $\dot\sigma(0) = \dot\sigma(L)$.

A.4. Let (p_n) be a Cauchy sequence in G/H. Then if d denotes the distance, $d(p_n, p_m) \to 0$ if $m, n \to \infty$. Let $B_\epsilon(o)$ be a relatively compact ball of radius $\epsilon > 0$ around the origin $o = \{H\}$ in G/H. Select N such that $d(p_N, p_m) < \frac{1}{2}\epsilon$ for $m \geqslant N$ and select $g \in G$ such that $g \cdot p_N = o$. Then $(g \cdot p_m)$ is a Cauchy sequence inside the compact ball $B_\epsilon(o)^-$, hence it, together with the original sequence, is convergent.

A.5. For $X \in \mathfrak{g}$ let $\tilde{X}$ denote the corresponding left invariant vector field on G. From Prop. 1.4 we know that (i) is equivalent to $\nabla_{\tilde Z}(\tilde{Z}) = 0$ for all $Z \in \mathfrak{g}$. But by (2), §9 in Chapter I this condition reduces to

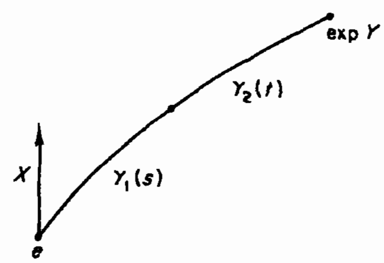
$$g(\tilde{Z}, [\tilde{X}, \tilde{Z}]) = 0 \qquad (X, Z \in \mathfrak{g})$$

which is clearly equivalent to (ii). Next (iii) follows from (ii) by replacing X by $X + Z$. But (iii) is equivalent to $\operatorname{Ad}(G)$-invariance of B so Q is right invariant. Finally, the map $J : x \to x^{-1}$ satisfies $J = R(g^{-1}) \circ J \circ L(g^{-1})$, so $dJ_g = dR(g^{-1})_e \circ dJ_e \circ dL(g^{-1})_g$. Since dJ_e is automatically an isometry, (v) follows.

A.6. Assuming first the existence of ∇, consider the affine transformation $\sigma : g \to \exp \frac{1}{2}Y g^{-1} \exp \frac{1}{2}Y$ of G which fixes the point $\exp \frac{1}{2}Y$ and maps γ_1, the first half of γ, onto the second half, γ_2. Since

$$\sigma = L(\exp \tfrac{1}{2}Y) \circ J \circ L(\exp -\tfrac{1}{2}Y),$$

we have $d\sigma_{\exp \frac{1}{2}Y} = -I$. Let $X^*(t) \in G_{\exp tY}$ $(0 \leqslant t \leqslant 1)$ be the family of vectors parallel with respect to γ such that $X^*(0) = X$. Then σ maps $X^*(s)$ along γ_1 into a parallel field along γ_2 which must be the field $-X^*(t)$ because $d\sigma(X^*(\frac{1}{2})) = -X^*(\frac{1}{2})$. Thus the map $\sigma \circ J = L(\exp \frac{1}{2}Y)\, R(\exp \frac{1}{2}Y)$ sends X into $X^*(1)$, as stated in part (i). Part (ii) now follows from Theorem 7.1, Chapter I, and part (iii) from Prop. 1.4.



Finally, we prove the existence of ∇. As remarked before Prop. 1.4, the equation $\nabla_{\tilde{X}}(\tilde{Y}) = \frac{1}{2}[\tilde{X}, \tilde{Y}]$ $(X, Y \in \mathfrak{g})$ defines uniquely a left invariant affine connection ∇ on G. Since $\tilde{X}^{R(g)} = (\mathrm{Ad}(g^{-1})X)^{\sim}$, we get

$$\nabla_{\tilde{X}^{R(g)}}(\tilde{Y}^{R(g)}) = \tfrac{1}{2}\{\mathrm{Ad}(g^{-1})[X, Y]\}^{\sim} = (\nabla_{\tilde{X}}(\tilde{Y}))^{R(g)};$$

this we generalize to any vector fields Z, Z' by writing them in terms of $\tilde{X}_i$ $(1 \leqslant i \leqslant n)$. Next

$$\nabla_{J\tilde{X}}(J\tilde{Y}) = J(\nabla_{\tilde{X}}(\tilde{Y})). \tag{1}$$

Since both sides are right invariant vector fields, it suffices to verify the equation at e. Now $J\tilde{X} = -\bar{X}$ where $\bar{X}$ is right invariant, so the problem is to prove

$$(\nabla_{\bar{X}}(\bar{Y}))_e = -\tfrac{1}{2}[X, Y].$$

For a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ we write $\mathrm{Ad}(g^{-1})Y = \sum_i f_i(g)X_i$. Since $\bar{Y}_g = dR(g)Y = dL(g)\,\mathrm{Ad}(g^{-1})Y$, it follows that $\bar{Y} = \sum_i f_i\tilde{X}_i$, so using ∇_2 and Lemma 4.2 from Chapter I, §4,

$$(\nabla_{\bar{X}}(\bar{Y}))_e = (\nabla_{\tilde{X}}(\bar{Y}))_e = \sum_i (Xf_i)_e\, X_i + \tfrac{1}{2} \sum_i f_i(e)[\tilde{X}, \tilde{X}_i]_e$$

Since $(Xf_i)(e) = \{(d/dt)\, f_i(\exp tX)\}_{t=0}$ and since

$$\left\{\frac{d}{dt}\,\mathrm{Ad}(\exp(-tX))(Y)\right\}_{t=0} = -[X, Y],$$

the expression on the right reduces to $-[X, Y] + \frac{1}{2}[X, Y]$, so (1) follows. As before, (1) generalizes to any vector fields Z, Z'.

The connection ∇ is the 0-connection of Cartan-Schouten [1].

B. The Exponential Mapping

B.1. At the end of §1 it was shown that $GL(2, R)$ has Lie algebra $\mathfrak{gl}(2, R)$, the Lie algebra of all 2×2 real matrices. Since $\det(e^{tX}) =$

$e^{t\,\mathrm{Tr}(X)}$, Prop. 2.7 shows that $\mathfrak{sl}(2, \boldsymbol{R})$ consists of all 2×2 real matrices of trace 0. Writing

$$X = a\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} + b\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} = \begin{pmatrix}a & b\\c & -a\end{pmatrix}$$

a direct computation gives for the Killing form

$$B(X, X) = 8(a^2 + bc) = 4\,\mathrm{Tr}(XX),$$

whence $B(X, Y) = 4\,\mathrm{Tr}(XY)$, and semisimplicity follows quickly. Part (i) is obtained by direct computation. For (ii) we consider the equation

$$e^X = \begin{pmatrix}\lambda & 0\\0 & \lambda^{-1}\end{pmatrix} \qquad (\lambda \in \boldsymbol{R},\quad \lambda \neq 1).$$

Case 1: $\lambda > 0$. Then $\det X < 0$. In fact $\det X = 0$ implies

$$I + X = \begin{pmatrix}\lambda & 0\\0 & \lambda^{-1}\end{pmatrix},$$

so $b = c = 0$, so $a = 0$, contradicting $\lambda \neq 1$. If $\det X > 0$, we deduce quickly from (i) that $b = c = 0$, so $\det X = -a^2$, which is a contradiction. Thus $\det X < 0$ and using (i) again we find the only solution

$$X = \begin{pmatrix}\log\lambda & 0\\0 & -\log\lambda\end{pmatrix}.$$

Case 2: $\lambda = -1$. For $\det X > 0$ put $\mu = (\det X)^{1/2}$. Then using (i) the equation amounts to

$$\cos\mu + (\mu^{-1}\sin\mu)a = -1, \qquad (\mu^{-1}\sin\mu)b = 0,$$
$$\cos\mu - (\mu^{-1}\sin\mu)a = -1, \qquad (\mu^{-1}\sin\mu)c = 0.$$

These equations are satisfied for

$$\mu = (2n+1)\pi \quad (n \in \boldsymbol{Z}), \qquad \det X = -a^2 - bc = (2n+1)^2\,\pi^2.$$

This gives infinitely many choices for X as claimed.

Case 3: $\lambda < 0$, $\lambda \neq -1$. If $\det X = 0$, then (i) shows $b = c = 0$, so $a = 0$; impossible. If $\det X > 0$ and we put $\mu = (\det X)^{1/2}$, (i) implies

$$\cos\mu + (\mu^{-1}\sin\mu)a = \lambda, \qquad (\mu^{-1}\sin\mu)b = 0,$$
$$\cos\mu - (\mu^{-1}\sin\mu)a = \lambda^{-1}, \qquad (\mu^{-1}\sin\mu)c = 0.$$

Since $\lambda \neq \lambda^{-1}$, we have $\sin \mu \neq 0$. Thus $b = c = 0$, so $\det X = -a^2$, which is impossible. If $\det X < 0$ and we put $\mu = (-\det X)^{1/2}$, we get from (i) the equations above with sin and cos replaced by sinh and cosh. Again $b = c = 0$, so $\det X = -a^2 = -\mu^2$; thus $a = \pm\mu$, so

$$\cosh \mu \pm \sinh \mu = \lambda, \qquad \cosh \mu \mp \sinh \mu = \lambda^{-1},$$

contradicting $\lambda < 0$. Thus there is no solution in this case, as stated.

B.3. Follow the hint.

B.4. Considering one-parameter subgroups it is clear that $\mathfrak{g}$ consists of the matrices

$$X(a, b, c) = \begin{pmatrix} 0 & c & 0 & a \\ -c & 0 & 0 & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (a, b, c \in \mathbf{R}).$$

Then $[X(a, b, c), \ X(a_1, b_1, c_1)] = X(cb_1 - c_1 b, c_1 a - ca_1, 0)$, so $\mathfrak{g}$ is readily seen to be solvable. A direct computation gives

$$\exp X(a, b, c) = \begin{pmatrix} \cos c & \sin c & 0 & c^{-1}(a \sin c - b \cos c + b) \\ -\sin c & \cos c & 0 & c^{-1}(b \sin c + a \cos c - a) \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus $\exp X(a, b, 2\pi)$ is the same point in G for all $a, b \in \mathbf{R}$, so exp is not injective. Similarly, the points in G with $\gamma = n2\pi$ $(n \in \mathbf{Z})$ $\alpha^2 + \beta^2 > 0$ are not in the range of exp. This example occurs in Auslander and MacKenzie [1]; the exponential mapping for a solvable group is systematically investigated in Dixmier [2].

B.5. Let N_0 be a bounded star-shaped open neighborhood of $0 \in \mathfrak{g}$ which exp maps diffeomorphically onto an open neighborhood N_e of e in G. Let $N^* = \exp(\frac{1}{2}N_0)$. Suppose S is a subgroup of G contained in N^*, and let $s \neq e$ in S. Then $s = \exp X$ $(X \in \frac{1}{2}N_0)$. Let $k \in \mathbf{Z}^+$ be such that $X, 2X, \ldots, kX \in \frac{1}{2}N_0$ but $(k + 1)X \notin \frac{1}{2}N_0$. Since N_0 is star-shaped, $(k + 1)X \in N_0$; but since $s^{k+1} \in N^*$, we have $s^{k+1} = \exp Y$, $Y \in \frac{1}{2}N_0$. Since exp is one-to-one on N_0, $(k + 1)X = Y \in \frac{1}{2}N_0$, which is a contradiction.

C. Subgroups and Transformation Groups

C.1. The proofs given in Lecture 26 for $SU^*(2n)$ and $Sp(n, C)$ generalize easily to the other subgroups.

C.2 Let G be a connnected commutative Lie group, (G*, p) its universal covering group (see Lecture 17 for definition). Then G* is topologically isomorphic to a Euclidean group $\mathbf{R}^n$. Thus G is topologically isomorphic to a factor group $\mathbf{R}^n/D$ where D is a discrete subgroup. By the theorem below for D this factor group is topologically isomorphic to $\mathbf{R}^q \times \mathbf{T}^m$ where T is the circle group. Thus by Theorem 2.6, G is analytically isomorphic to $\mathbf{R}^q \times \mathbf{T}^m$.

For the last statement let $\bar{\gamma}$ be the closure of γ in H. By the first statement and Theorem 2.3, $\bar{\gamma} = R^n \times T^m$ for some n, $m \in Z^+$. But γ is dense in $\bar{\gamma}$, so either $n = 1$ and $m = 0$ (γ closed) or $n = 0$ ($\bar{\gamma}$ compact).

Theorem. *Let V be a vector space over* $\mathbf{R}$ *and $D \subset V$ a discrete subgroup. Then there exist linearly independent vectors $v_1, \ldots, v_r$ in V such that*

$$D = \sum_1^r \mathbf{Z} v_i\,.$$

Proof. We may assume that D spans V and shall prove the result by induction on $r = \dim V$. Consider an indivisible element $d_0 \in D$ (i.e., $td_0 \in D$, $0 < t \leq 1 \Rightarrow t = 1$). Let U be the line $\mathbf{R}d_0$, W a complementary subspace and $V' = V/U$. We have $D \cap U = \mathbf{Z}d_0$ because of the choice of d_0. The natural mapping $\pi : V \to V'$ gives a homeomorphism of W onto V'. Let $D' = \pi(D)$. We claim D' is discrete. Otherwise O would be a limit point of D' in V' so there would be a sequence $(w_n) \subset W$, $(w_n \neq 0)$ such that $\pi(w_n)$ is a sequence in D' converging to O in V'. Let $d_n \in D$ be such that $\pi(d_n) = \pi(w_n)$. Then $d_n - w_n \in U$ and $w_n \to O$. Select $z_n \in D \cap U$ such that $d_n - w_n - z_n$ (which belongs to U) lies between O and d_0. Then passing to a subsequence we may assume $d_n - w_n - z_n$ converges to a limit $d^* \in U$. Then $d_n - z_n \to d^*$ and since $d_n, z_n \in D$ we have $d_n = z_n + d^*$ for sufficiently large n. But $z_n \in U$ so $d_n \in U$ for such n and this contradicts $\pi(d_n) = \pi(w_n) \neq 0$.

Thus D' is discrete in V' so by the inductive hypothesis,

$$D' = \sum_1^{r-1} \mathbf{Z} v'_i$$

for a suitable basis (v'_i) of V'. Select $v_i \in D$ such that $\pi(v_i) = v'_i$ $(1 \leq i \leq r-1)$. If $d \in D$ then $\pi(d) = \sum_1^{r-1} n_i v'_i$ so $d - \sum_1^{r-1} n_i v_i \in D \cap U = \mathbf{Z}d_0$ so the result follows with $v_r = d_0$. □

C.3. By Theorem 2.6, I is analytic and by Lemma 1.12, dI is injective.
Q.E.D.

C.4. The mapping ψ_g turns $g \cdot N_0$ into a manifold which we denote by $(g \cdot N_0)_x$. Similarly, $\psi_{g'}$ turns $g' \cdot N_0$ into a manifold $(g' \cdot N_0)_y$. Thus we have two manifolds $(g \cdot N_0 \cap g' \cdot N_0)_x$ and $(g \cdot N_0 \cap g' \cdot N_0)_y$ and must show that the identity map from one to the other is analytic. Consider the analytic section maps

$$\sigma_g : (g \cdot N_0)_x \to G, \qquad \sigma_{g'} : (g' \cdot N_0)_y \to G$$

defined by

$$\sigma_g(g \exp(x_1X_1 + \ldots + x_rX_r) \cdot p_0) = g \exp(x_1X_1 + \ldots + x_rX_r),$$
$$\sigma_{g'}(g' \exp(y_1X_1 + \ldots + y_rX_r) \cdot p_0) = g' \exp(y_1X_1 + \ldots + y_rX_r),$$

and the analytic map

$$J_g : \pi^{-1}(g \cdot N_0) \to (g \cdot N_0)_x \times H$$

given by

$$J_g(z) = (\pi(z), [\sigma_g(\pi(z))]^{-1}z).$$

Furthermore, let $P : (g \cdot N_0)_x \times H \to (g \cdot N_0)_x$ denote the projection on the first component. Then the identity mapping

$$I : (g \cdot N_0 \cap g' \cdot N_0)_y \to (g \cdot N_0 \cap g' \cdot N_0)_x$$

can be factored:

$$(g \cdot N_0 \cap g' \cdot N_0)_y \xrightarrow{\sigma_{g'}} \pi^{-1}(g \cdot N_0) \xrightarrow{J_g} (g \cdot N_0)_x \times H \xrightarrow{P} (g \cdot N_0)_x.$$

In fact, if $p \in g \cdot N_0 \cap g' \cdot N_0$, we have

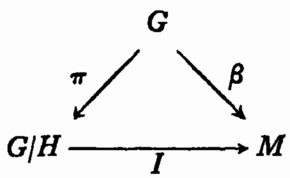
$$p = g \exp(x_1 X_1 + \dots + x_r X_r) \cdot p_0 = g' \exp(y_1 X_1 + \dots + y_r X_r) \cdot p_0,$$

so for some $h \in H$,

$$\begin{aligned} P(J_g(\sigma_{g'}(p))) &= P(J_g(g' \exp(y_1 X_1 + \dots + y_r X_r))) \\ &= P(\pi(g' \exp(y_1 X_1 + \dots + y_r X_r)), h) \\ &= P(\pi(g \exp(x_1 X_1 + \dots + x_r X_r)), h) \\ &= g \exp(x_1 X_1 + \dots + x_r X_r)) \cdot p_0. \end{aligned}$$

Thus I is composed of analytic maps so is analytic, as desired.

C.5. The subgroup $H = G_p$ of G leaving p fixed is closed, so G/H is a manifold. The map $I : G/H \to M$ given by $I(gH) = g \cdot p$ gives a bijection of G/H onto the orbit $G \cdot p$. Carrying the differentiable structure over on $G \cdot p$ by means of I, it remains to prove that $I : G/H \to M$ is everywhere regular. Consider the maps on the diagram



where $\pi(g) = gH$, $\beta(g) = g \cdot p$ so $\beta = I \circ \pi$. If we restrict π to a local cross section, we can write $I = \beta \circ \pi^{-1}$ on a neighborhood of the origin in G/H. Thus I is C^∞ near the origin, hence everywhere. Moreover, the map $d\beta_e : \mathfrak{g} \to M_p$ has kernel $\mathfrak{h}$, the Lie algebra of H (cf. proof of Prop. 4.3). Since $d\pi_e$ maps $\mathfrak{g}$ onto $(G/H)_H$ with kernel $\mathfrak{h}$ and since $d\beta_e = dI_H \circ d\pi_e$, wee that dI_H is one-to-one. Finally, if $T(g)$ denotes the diffeomorphism $m \to g \cdot m$ of M, we have $I = T(g) \circ I \circ \tau(g^{-1})$, whence

$$dI_{gH} = dT(g)_p \circ dI_H \circ d\tau(g^{-1})_{gH},$$

so I is everywhere regular.

C.6. By local connectedness each component of G is open. It acquires an analytic structure from that of G_0 by left translation. In order to show the map $\varphi : (x, y) \to xy^{-1}$ analytic at a point $(x_0, y_0) \in G \times G$ let G_1 and G_2 denote the components of G containing x_0 and y_0, respectively. If $\varphi_0 = \varphi \mid G_0 \times G_0$ and $\psi = \varphi \mid G_1 \times G_2$, then

$$\psi = L(x_0 y_0^{-1}) \circ I(y_0) \circ \varphi_0 \circ L(x_0^{-1}, y_0^{-1}),$$

where $I(y_0)(x) = y_0 x y_0^{-1}$ $(x \in G_0)$. Now $I(y_0)$ is a continuous automorphism of the Lie group G_0, hence by Theorem 2.6, analytic; so the expression for ψ shows that it is analytic.

C.8. If N with the indicated properties exists we may, by translation, assume it passes through the origin $o = \{H\}$ in M. Let L be the subgroup $\{g \in G : g \cdot N = N\}$. If $g \in G$ maps o into N, then $gN \cap N \neq \emptyset$; so by assumption, $gN = N$. Thus $L = \pi^{-1}(N)$ where $\pi : G \to G/H$ is the natural map. Using Theorem 15.5, Chapter I we see that L can be given the structure of a submanifold of G with a countable basis and by the transitivity of G on M, $L \cdot o = N$. By C.7, L has the desired property. For the converse, define $N = L \cdot o$ and use Prop. 4.4 or Exercise C.5. Clearly, if $gN \cap N \neq \emptyset$, then $g \in L$, so $gN = N$.

For more information on the primitivity notion which goes back to Lie see e.g. Golubitsky [1].

D. Closed Subgroups

D.1. R^2/Γ is a torus (Exercise C.2), so it suffices to take a line through 0 in R^2 whose image in the torus is dense.

D.2. $\mathfrak{g}$ has an $\mathrm{Int}(\mathfrak{g})$-invariant positive definite quadratic form Q. The proof of Prop. 6.6 now shows $\mathfrak{g} = \mathfrak{z} + \mathfrak{g}'$ ($\mathfrak{z}$ = center of $\mathfrak{g}$, $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ compact and semisimple). The groups $\mathrm{Int}(\mathfrak{g})$ and $\mathrm{Int}(\mathfrak{g}')$ are analytic subgroups of $GL(\mathfrak{g})$ with the same Lie algebra so coincide.

D.3. We have

$$\alpha_{0,\frac{1}{3}}(c_1, c_2, s) = (c_1, e^{2\pi i/3}c_2, s)$$

$$\begin{aligned}(a_1, a_2, r)(c_1, c_2, s)(a_1, a_2, r)^{-1} \\ = (a_1(1 - e^{2\pi i s}) + c_1 e^{2\pi i r}, a_2(1 - e^{2\pi i h s}) + c_2 e^{2\pi i h r}, s)\end{aligned}$$

so $\alpha_{0,\frac{1}{3}}$ is not an inner automorphism, and $A_{0,\frac{1}{3}} \notin \mathrm{Int}(\mathfrak{g})$. Now let $s_n \to 0$ and let $t_n = hs_n + hn$. Select a sequence $(n_k) \subset Z$ such that $hn_k \to \frac{1}{3}$ (mod 1) (Kronecker's theorem), and let τ_k be the unique point in $[0, 1)$ such that $t_{n_k} - \tau_k \in Z$. Putting $s_k = s_{n_k}$, $t_k = t_{n_k}$, we have

$$\alpha_{s_k, t_k} = \alpha_{s_k, \tau_k} \to \alpha_{0,\frac{1}{3}}.$$

Note: G is a subgroup of $H \times H$ where $H = \begin{pmatrix} 1 & 0 \\ c & \alpha \end{pmatrix}$, $c \in C$, $|\alpha| = 1$.

E. Invariant Differential Forms

E.1. The affine connection on G given by $\nabla_{\tilde{X}}(\tilde{Y}) = \frac{1}{2}[\tilde{X}, \tilde{Y}]$ is torsion free; and by (5), §7, Chapter I, if ω is a left invariant 1-form,

$$\nabla_{\tilde{X}}(\omega)(\tilde{Y}) = -\omega(\nabla_{\tilde{X}}(\tilde{Y})) = -\tfrac{1}{2}\omega(\theta(\tilde{X})(\tilde{Y})) = \tfrac{1}{2}(\theta(\tilde{X})\omega)(\tilde{Y}),$$

so $\nabla_{\tilde{X}}(\omega) = \frac{1}{2}\theta(\tilde{X})(\omega)$ for all left invariant forms ω. Now use Exercise C.4 in Chapter I.

E.2. The first relation is proved as (4), §7. For the other we have $g{}^tg = I$, so $(dg){}^tg + g{}^t(dg) = 0$. Hence $(g^{-1}\,dg) + {}^t(dg)({}^tg)^{-1} = 0$ and $\Omega + {}^t\Omega = 0$.

For $U(n)$ we find similarly for $\Omega = g^{-1}\,dg$,

$$d\Omega + \Omega \wedge \Omega = 0, \qquad \Omega + {}^t\bar{\Omega} = 0.$$

For $Sp(n) \subset U(2n)$ we recall that $g \in Sp(n)$ if and only if

$$g{}^t\bar{g} = I_{2n}, \qquad gJ_n{}^tg = J_n$$

(cf. Chapter X). Then the form $\Omega = g^{-1}\,dg$ satisfies

$$d\Omega + \Omega \wedge \Omega = 0, \qquad \Omega + {}^t\bar{\Omega} = 0, \qquad \Omega J_n + J_n{}^t\Omega = 0.$$

E.3. A direct computation gives

$$g^{-1}\,dg = \begin{pmatrix} 0 & dx & dz - x\,dy \\ 0 & 0 & dy \\ 0 & 0 & 0 \end{pmatrix}$$

and the result follows.

F. Invariant Measures

F.1. (i) If H is compact, $|\det(\mathrm{Ad}_G(H))|$ and $|\det(\mathrm{Ad}_H(H))|$ are compact subgroups of the multiplicative groups of the positive reals, hence identically 1.

(ii) G/H has an invariant measure so $|\det \mathrm{Ad}_H(h)| = |\det \mathrm{Ad}_G(h)|$, which by unimodularity of G equals 1.

(iii) Let $G_0 = \{g \in G : |\det \mathrm{Ad}_G(g)| = 1\}$. Then G_0 is a normal subgroup of G containing H. Since $\mu(G/H) < \infty$, Prop. 1.13 shows that the group G/G_0 has finite Haar measure, and hence is compact. Thus the image $|\det \mathrm{Ad}_G(G)|$ is a compact subgroup of the group of positive reals, and hence consists of 1 alone.

F.2. The element $H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ spans the Lie algebra $\mathfrak{o}(2)$ and $\exp \mathrm{Ad}(g)tH = g \exp tHg^{-1} = \exp(-tH)$.

F.3. We have $\det \mathrm{Ad}(\exp X) = \det(e^{\mathrm{ad}\,X}) = e^{\mathrm{Tr}(\mathrm{ad}\,X)}$, so (i) follows. For (ii) we know that G/H has an invariant measure if and only if

$$\exp(\mathrm{Tr}(\mathrm{ad}_{\mathfrak{g}}\,T)) = \exp(\mathrm{Tr}(\mathrm{ad}_{\mathfrak{h}}\,T)), \quad T \in \mathfrak{h}.$$

Put $T = tX_i$ $(r < i \leq n)$, $t \in \mathbf{R}$, and differentiate with respect to t. Then the desired relations follow.

F.4. To each $g \in M(n)$ we associate the translation T_x by the vector $x = g \cdot o$ and the rotation k given by $g = T_x k$. Then $kT_x k^{-1} = T_{k \cdot x}$, so

$$g_1 g_2 = T_{x_1} k_1 T_{x_2} k_2 = T_{x_1 + k_1 \cdot x_2} k_1 k_2 .$$

Since $g_{k_1, x_1} \cdot g_{k_2, x_2} = g_{k_1 k_2, x_1 + k_1 \cdot x_2}$ this shows that the mapping $g \to g_{k,x}$ is an isomorphism. Also

$$\int f(g_{k_0, x_0} g_{k, x}) \, dk \, dx = \int f(g_{k_0 k, \, x_0 + k_0 \cdot x}) \, dk \, dx$$

$$= \int f(g_{k, x}) \, dk \, dx$$

since dx is invariant under $x \to x_0 + k_0 \cdot x$.

F.5. By [DS], Chapter II, §7, the entries ω_{ij} in the matrix $\Omega = X^{-1} \, dX$ constitute a basis of the Maurer–Cartan forms (the left invariant 1-forms) on $GL(n, \mathbf{R})$. Writing $dX = X\Omega$ we obtain from [DS] (Chapter I, §2, No. 3) for the exterior products

$$\prod_{i,j} dx_{ij} = (\det X)^n \prod_{i,j} \omega_{ij},$$

so $|\det X|^{-n} \prod_{i,j} dx_{ij}$ is indeed a left invariant measure. The same result would be obtained from the right invariant matrix $(dX)X^{-1}$ so the unimodularity follows.

F.6. Let the subset $G' \subset G$ be determined by the condition $\det X_{11} \neq 0$ and define a measure $d\mu$ on G' by

$$d\mu = |\det X_{11}|^{-1} \prod_{(i,j) \neq (1,1)} dx_{ij}.$$

If dg is a bi-invariant Haar measure on G we have (since $G - G'$ is a null set)

$$\int_G f(g) \, dg = \int_{G'} f(g) \, dg = \int_{G'} f(g) J(g) \, d\mu,$$

where J is a function on G'. Let T be a diagonal matrix with $\det T = 1$ and $t_1, \ldots, t_n$ its diagonal entries. Under the map $X \to TX$ the product $\prod_{(i,j) \neq (1,1)} dx_{ij}$ is multiplied by $t_1^{n-1} t_2^n \cdots t_n^n$ and $|\det X_{11}|$ is multiplied by $t_2 t_3 \cdots t_n$. Since $\det T = 1$, these factors are equal, so the set G' and the measure μ are preserved by the map $X \to TX$. If A is a supertriangular matrix with diagonal 1, the mapping $X \to AX$ is supertriangular with diagonal 1 if the elements x_{ij} are ordered lexicographically. Thus $\prod_{(i,j) \neq (1,1)} dx_{ij}$ is unchanged and a simple inspection shows† $\det((AX)_{11}) = \det(X_{11})$. It follows that G' and $d\mu$ are invariant under each map $X \to UX$ where U is a supertriangular matrix in G. By transposition, G' and $d\mu$ are invariant under the map $X \to XV$ where V is a lower triangular matrix in G. The integral formulas above therefore show that $J(UXV) \equiv J(X)$. Since the products UV form a dense subset of G ([DS], Chapter IX, Exercise A2) μ is a constant multiple of dg. For

F.7. A simple computation shows that the measures are invariant under multiplication by diagonal matrices as well as by unipotent matrices; hence they are invariant under $\mathbf{T}(n, \mathbf{R})$.

†Note in fact that $(AX)_{11} = A_{11} X_{11}$ if the x_{ij} are ordered by $x_{11}, \ldots, x_{1n}, x_{2n}, \ldots x_{2n}, \ldots x_{n1}, \ldots x_{nn}$.

G. Compact Real Forms and Complete Reducibility

G.1. Since the Killing form of $\mathfrak{g}$ is nondegenerate, there exists a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$ such that

$$B(Z, Z) = -\sum_1^n z_i^2 \qquad \text{if} \quad Z = \sum_1^n z_i e_i \qquad (1)$$

Let the structural constants $c_{ijk} \in C$ be determined by

$$[e_i, e_j] = \sum_1^n c_{ijk} e_k$$

Then

$$B(Z, Z) = \text{Tr}(\text{ad } Z \text{ ad } Z) = \sum_{i,j} \left(\sum_{h,k} c_{ikh} c_{jhk} \right) z_i z_j$$

so

$$\sum_{h,k} c_{ikh} c_{jhk} = -\delta_{ij}$$

Also,

$$B([X_i, X_j], X_k) + B(X_j, [X_i, X_k]) = 0$$

so

$$c_{ijk} + c_{ikj} = 0$$

and

$$\sum_{i,h,k} c_{ihk}^2 = n$$

The space

$$\mathfrak{u} = \sum_1^n Re_i$$

is a real form of $\mathfrak{g}$ if and only if all the c_{ijk} are real.

Consider now the set $\mathfrak{F}$ of all bases $(e_1, \ldots, e_n)$ of $\mathfrak{g}$ such that (1) holds. Consider the function f on $\mathfrak{F}$ given by

$$f(e_1, \ldots, e_n) = \sum_{i,j,k} |c_{ijk}|^2$$

Then we have seen that

$$\sum_{i,j,k} |c_{ijk}|^2 \geqslant \left| \sum_{i,j,k} c_{ijk}^2 \right| = \sum_{i,j,k} c_{ijk}^2 = n$$

and the equality sign holds if and only if all the c_{ijk} are real, that is, if and only if

$$\mathfrak{u} = \sum_1^n Re_i$$

is a real form. In this case it is a compact real form in view of (1) and **Lemma 6.1**

Thus Theorem 6.3 follows if one can prove: (I) The function f on $\mathfrak{F}$ has a minimum value; and (II) this minimum value is attained at a point $(e_1^0, \ldots, e_n^0) \in \mathfrak{F}$ for which the structural constants are real. Note that (II) is equivalent to (II′): The minimum of f is n.

B.3. (i) Suppose first V is real. Since a compact group of linear transformations of V leaves invariant a positive definite quadratic form, this part follows (as Prop. 6.6 in Chapter II) by orthogonal complementation. If V is complex, we use a positive definite Hermitian form instead.

For (ii) we suppose first V is complex. Then π extends to a representation of the complexification $\mathfrak{g}^C$ on V. Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}^C$, U the (compact) simply connected Lie group with Lie algebra $\mathfrak{u}$, and extend π to a representation of U on V, also denoted π. If $W \subset V$ is $\pi(\mathfrak{g})$-invariant, it is also $\pi(\mathfrak{g}^C)$- and $\pi(U)$-invariant and a $\pi(U)$-invariant complementary subspace will also be $\pi(\mathfrak{g}^C)$-invariant. Finally, we consider the case when V is real using a trick from Freudenthal and de Vries [1], §35. We view π as a representation of $\mathfrak{g}$ on the complexification V^c of V and then each member of $\pi(\mathfrak{g})$ commutes with the conjugation σ of V^c with respect to V. Let $W \subset V$ be a $\pi(\mathfrak{g})$-invariant subspace. Then the complexification $W^c = W + iW$ is a $\pi(\mathfrak{g})$-invariant subspace of V^c, so by the first case W^c has a $\pi(\mathfrak{g})$-invariant complement $Z' \subset V^c$. Let $Z = (1 + \sigma)(Z' \cap (1 - \sigma)^{-1}(iW))$. Since $\sigma(1 + \sigma) = \sigma + 1$ and $\pi(X)\sigma = \sigma\pi(X)$ $(X \in \mathfrak{g})$, we have $Z \subset V$, $\pi(\mathfrak{g})Z \subset Z$. Also $Z \cap W = \{0\}$. In fact, if $z \in Z \cap W$, there exists a $z' \in Z'$ such that $(1 - \sigma)z' \in iW$, $(1 + \sigma)z' = z$. Hence $z' = \frac{1}{2}(1 - \sigma)z' + \frac{1}{2}(1 + \sigma)z' \in W^c$, so $z' = 0$ and $z = 0$. Finally, $W + Z = V$. In fact, if $v \in V$, then $v = w' + z'$ $(w' \in W^c, z' \in Z')$. Then $w' + z' = v = \sigma v = \sigma w' + \sigma z'$, so $(1 - \sigma)z' = (1 - \sigma)(-w') \in iW$, so $z' \in Z' \cap (1 - \sigma)^{-1}(iW)$ and $(1 + \sigma)z' \in Z$. Hence $v = \frac{1}{2}(1 + \sigma)w' + \frac{1}{2}(1 + \sigma)z' \in W + Z$.

(This "theorem of complete reducibility" was first proved by H. Weyl [1], I, §5 by a similar method; algebraic proofs were later found by Casimir and van der Waerden [1] and by Whitehead [4].)