Sophus Lie and the Role of Lie Groups in Mathematics

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As Lie group theory has developed it has also become more and more pervasive in its influence on other mathematical disciplines. The original founder of this theory was a Norwegian, Marius Sophus Lie, who was born in Nordfjordeid, 1842. In order to understand the background and motivation to Lie's work we must go further back.

A central problem in algebra at the end of the 18^{th} century was that of solving algebraic equations. While 2^{nd}, 3^{rd} and 4^{th} degree equations could be solved explicitly by radicals it was suspected, particularly through the work of Lagrange (1771), that the general 5^{th} degree equation could not be solved in this way. Ruffini, in 1813, proposed a proof of this; however the proof was generally found to be unsatisfactory. Abel gave another proof in 1824 which after subsequent repairs has been considered complete. But to Galois belongs the far-reaching idea (around 1830) of attaching to the equation a certain finite permutation group (of the roots), now called the Galois group. A remarkable theorem in Galois theory states that the solvability of this group is equivalent to the solvability of the equation by radicals. The equation $x^5 - x - \frac{1}{3} = 0$ has Galois group S_5, the symmetric group of five letters which is not solvable; thus the Ruffini-Abel result follows.

When Sylow gave a lecture on these matters at the University of Oslo 1863, a farmer's son, by the name of Sophus Lie (1842–1899), was in the audience. Although his interests were oriented more towards Geometry than Algebra, Galois' ideas made a great impression on him. After his friendly and productive collaboration with Klein 1970–71, Lie conceived the idea of developing an analog for differential equations to Galois theory for algebraic equations. I shall try to explain the foundations of this theory.

The differential equation

$$\frac{dy}{dx} = \frac{Y(x,y)}{X(x,y)} \tag{1}$$

is called *stable* under a transformation T of the plane if T permutes the integral curves.

Example.

$$\frac{dy}{dx} = f(x)$$

is stable under each transformation

$$T_t : \begin{array}{l} x \longrightarrow x \\ y \longrightarrow y + t \end{array}$$

Lie's theorem [9]. *Suppose* (1) *is stable under a one-parameter group*

$$T_t : \mathbf{R}^2 \longrightarrow \mathbf{R}^2 \qquad (T_{t+s} = T_t \circ T_s).$$

Put

$$\Phi_p = \left\{ \frac{d(T_t \cdot p)}{dt} \right\}_{t=0} = (\xi(p), \eta(p)) = \xi(p)\frac{\partial}{\partial x} + \eta(p)\frac{\partial}{\partial y}.$$

Then $(X\eta - Y\xi)^{-1}$ *is an integrating factor to the equation* $X dy - Y dx = 0$, *that is* $\frac{X dy - Y dx}{X\eta - Y\xi}$ *is a total differential.*

Example.

$$\frac{dy}{dx} = \frac{y + x(x^2 + y^2)}{x - y(x^2 + y^2)}.$$

This differential equation can be written

$$\frac{\frac{dy}{dx} - \frac{y}{x}}{1 + \frac{y}{x}\frac{dy}{dx}} = x^2 + y^2.$$

The left hand side is the tangent of the angle between the integral curve and the radius vector; thus the integral curves intersect each circle $x^2 + y^2 = r^2$ under a fixed angle. Each rotation around the origin permutes the integral curves, i.e., leaves the equation stable.

Lie's theorem can be used on the one-parameter group

$$T_t : (x, y) \longrightarrow (x \cos t - y \sin t, x \sin t + y \cos t).$$

We find by differentiating with respect to t and putting $t = 0$, $\Phi = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, and the theorem gives the solution $y = x\tan(\frac{1}{2}(x^2 + y^2) + C)$.

Generalizing T_t in Lie's theorem let

$$T_{(t)} : x_i' = f_i(x_1, \ldots, x_n; t_1, \ldots t_r)$$

be an r-parameter group of transformations of n-space ([1], [12], [6], p. 144). This means that the f_i are C^∞ functions, $T_{(0)} = I$ and $T_{(t)} \circ T_{(s)}^{-1} = T_{(u)}$, where $u = (u_1, \dots u_r)$ depends analytically on (t) and (s). Generalizing Φ above, Lie introduced the vector fields ("infinitesimal transformations")

$$\Phi_k = \sum_{i=1}^{n} \left[\frac{\partial f_i}{\partial t_k}\right]_{(t)=0} \frac{\partial}{\partial x_i}$$

and, as a result of the group property of $T_{(t)}$, proved the fundamental relation, ([6], p. 146):

$$\Phi_k \circ \Phi_\ell - \Phi_\ell \circ \Phi_k = \sum_{p=1}^{r} c^p_{k\ell} \Phi_p,$$

where the $c^p_{k\ell}$ are *constants* satisfying

$$(2) \qquad c^p_{k\ell} = -c^p_{\ell k}, \qquad \sum_{q=1}^{r} (c^\ell_{kq} c^q_{\ell m} + c^p_{mq} c^q_{k\ell} + c^p_{\ell q} c^q_{mk}) = 0.$$

Along with his students in Leipzig where Lie was a professor 1886–1898, Lie embarked on the project of classifying all groups $T_{(t)}$, i.e., all possibilities for the functions f_i. The motivation was application to partial differential equations in the spirit of Lie's theorem above. While the classification succeeded only for small n, the effort resulted in several important results connecting the (f_i) and the $(c^p_{k\ell})$ as well as in extensive integration theories for partial differential equations.

At this time (around 1880) an outsider to Lie's group, Wilhelm Killing, began working on the purely algebraic problem of classifying all constants $c^p_{k\ell}$ satisfying (2). Killing was motivated by geometry rather than by potential applications to differential equations. The problem is best explained in terms of the concept of a Lie algebra. This is a vector space V with a bilinear rule of composition $[X, Y]$ satisfying the relations

$$[X, Y] = -[Y, X], \qquad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0,$$

which are equivalent to (2). The simplest nontrivial example of a Lie algebra is the vector space of $n \times n$ matrices with $[A, B] = AB = BA$. Killing's problem was then to classify all Lie algebras up to isomorphism. Assuming V finite-dimensional over C, simple (that is there is no proper subspace $W \neq 0$ satisfying $[V, W] \subset W$)

Killing found the remarkable result [10], that apart from the four classical infinite series A_n, B_n, C_n, D_n of Lie algebras there are just five more, denoted E_6, E_7, E_8, F_4 and G_2 of dimensions $78, 133, 248, 52$ and 14 respectively. It is a testimony to the greatness of this discovery that even today when the subject is 100 years old the result comes as a complete surprise. Killing's starting point was to study the linear transformations $A_X : Y \longrightarrow [X, Y]$ by means of their eigenvalues. His proofs had many gaps and errors but in his thesis [2], Elie Cartan managed to complete the proofs.

It is not usually a particularly creditable accomplishment to give a wrong proof of a theorem which has been generally suspected for some time. But when an incomplete proof leads to a surprise on the level of Killing's discovery of the exceptional groups, more tolerance is called for.

Although Cartan's dissertation was clearly and carefully written it did not attract many other mathematicians to the field. Cartan therefore had the topic of Lie algebra theory pretty much for himself during the next 25 years. Lie died in 1899 at the age of 56, long before the influence of Lie algebras in mathematics was realized.

Through Cartan's work, Lie algebra theory acquired an existence on its own, independent of differential equations where it originated. Recently, however, Lie groups have had an increasing impact on many fields in mathematis. I will now give some examples, starting with the field where Lie group theory originated.

Partial differential equations. Gauss's mean value theorem says that a function $u(x, y)$ satisfies Laplace's equation

$$\Delta u = 0$$

if and only if

$$\frac{1}{2\pi} \int_0^{2\pi} u(x_1 + r\cos\theta, x_2 + r\sin\theta) d\theta = u(x_1, x_2).$$

Note that the left hand side is the average of the values of u on a circle with center (x_1, x_2) and radius r.

If we look carefully we see that there is a group at work here, namely the group G of rigid motions in the plane. Let x denote the translation by (x_1, x_2). Let y

denote the translation by $(r, 0)$. Let k denote the rotation by θ and K the group of all such rotations around 0. Then the formula reads

$$\int_K u(xky \cdot 0)dk = u(x \cdot 0) \qquad \text{all } x, y \in G.$$

Also the polynomials $P(\Delta)$ can be characterized as the differential operators on $\mathbf{R}^2$, which are invariant under all isometries of $\mathbf{R}^2$. Thus the mean value theorem can be stated: The integral formula holds if and only if $Du = 0$ for all invariant differential operators D without constant term. The advantage of this formulation (which is due to Godement; see [4] or [8], p. 403) is that now the theorem holds for all Riemannian manifolds with a transitive group G of isometries, K being the subgroup of G leaving certain point fixed and dk being the normalized Haar measure.

Let me also recall Schwarz theorem for the Dirichlet problem of constructing a harmonic function on $|z| < 1$ with boundary values given by a continuous function F on $|z| = 1$.

Let

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} F(e^{i\phi})d\phi.$$

Theorem. (Schwartz) $\lim_{r \longrightarrow 1} u(re^{i\theta}) = F(e^{i\theta})$.

We can assume $\theta = 0$. The usual proof considers

$$u(r) - F(1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos\phi + r^2} (F(e^{i\phi}) - F(1))d\phi$$

and splits

$$\int_{-\pi}^{\pi} = \int_{-\delta}^{\delta} + \int_{-\pi}^{\delta} + \int_{\delta}^{\pi}.$$

The first term is then made small by the continuity of F and the other terms can be made small by taking r close to 1. But the group theory gives here a much simpler and more conceptual proof ([8], p. 69).

We have

$$u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\phi})d\phi.$$

If g is any conformal tranformation of $|z| < 1$ onto itself then

$$u(g \cdot 0) = \frac{1}{2\pi} \int_0^{2\pi} F(g \cdot e^{i\phi}) d\phi.$$

In particular, take $g_t(z) = \frac{\cosh tz + \sinh t}{\sinh tz + \cosh t}$. Since

$$g_t \cdot 0 = \tanh t, \qquad g_t(e^{i\phi}) = \frac{e^{i\phi} + \tanh t}{\tanh t e^{i\phi} + 1},$$

we get

$$\lim_{r \longrightarrow 1} u(r) = \lim_{t \longrightarrow \infty} u(g_t \cdot 0) = \frac{1}{2\pi} \int_0^{2\pi} \lim_{t \longrightarrow \infty} F[\frac{e^{i\phi} + \tanh t}{\tanh t e^{i\phi} + 1}] d\phi$$
$$= F(1).$$

Number theory. While abelian groups and their characters enter in number theory at an early and elementary stage, the appearance of non abelian Lie groups is relatively recent.

Let $A(n)$ denote the number of solutions of the equation $n = x_1^2 + \cdots + x_4^2$ in integers. Following Hecke we associate with A the function

$$\phi(z) = \sum_0^\infty e^{\pi i n z} A(n),$$

which is holomorphic in the upper half plane. The group $SL(2, \mathbf{R})$ of 2×2 real matrices operates in the upper half plane by $z \longrightarrow \frac{az+b}{cz+d}$. Clearly ϕ satisfies

$$\phi(z+2) = \phi(z)$$

and using Poisson summation formula in Fourier Analysis one finds the subtler formula

$$\phi(-\frac{1}{z}) = z^2 \phi(z).$$

These relations can be combined in

$$\phi(\frac{az+b}{cz+d}) = (cz+d)^2 \phi(z)$$

for all $\binom{ab}{cd}$ in a certain subgroup Γ of finite index in $SL(2, \mathbf{Z})$. Such functions ϕ are called automorphic forms; they have an extensive theory which then implies precise information about $A(n)$.

A famous automorphic form is the function $\eta(e^{2\pi i z})$ where

$$\eta(X) = X^{\frac{1}{24}} \prod_1^\infty (1 - X^n)$$

is the Dedekind η-function. The function $\phi(X) = \prod_1^\infty (1 - X^n)$ was shown by Euler (1748) to have the expansion

$$\phi(X) = \sum_{-\infty}^{\infty} (-1)^k X^{(3k^2+k)/2}.$$

Later (1828) Jacobi obtained the expansion

$$\phi(X)^3 = \sum_{-\infty}^{\infty} (4k+1) X^{2k^2+k}.$$

Dyson proved recently similar formulas for a certain infinite set of powers of $\phi(X)$; but Lie groups were not on his mind at the time so he did not recognize these powers as the dimensions of the classical Lie algebras. Macdonald did, and proved, by means of Lie group theory such a formula for $\phi^{\dim(\mathfrak{g})}$ for each complex simple Lie algebra $\mathfrak{g}$. Kač, Lepowsky and others have carried this much further and have obtained many new relationships between Lie algebra theory and formulas from combinatorial number theory.

Physics. Here the role of Lie groups is extremely prominent as one can see by looking at recent issues of journals on theoretical physics. Let me mention just two examples of significant progress in physics suggested by Lie group theory.

It is a consequence of Maxwell's equations that electric and magnetic fields change in space and time according to the wave equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$$

where c is the velocity of light. The group of linear transformations which leaves invariant the quadratic form

$$x^2 + y^2 + z^2 - c^2 t^2$$

is called the Lorentz group. This group together with the translations

$$(x, y, z, t) \longrightarrow (x + \alpha, y + \beta, z + \gamma, t + \tau) \qquad (\alpha, \beta, \gamma, \delta \text{ const.})$$

generate the so-called inhomogeneous Lorentz group. The inhomogeneous Lorentz group leaves the wave equation invariant. This implicit physical significance of the Lorentz group is made explicit in the special theory of relativity where its elements are interpreted in terms of pure mechanics. One can conversely prove that the only differential operators on $\mathbf{R}^4$ invariant under the inhomogeneous Lorentz group are polynomials in the wave operator. Dirac wanted to set up relativistically invariant equations for the motion of the electron and at the same time have only first order time derivatives appear as dictated by Quantum Mechanics. By the remark above this is impossible by a differential operator on scalar-valued functions. So instead Dirac considered vector functions $\psi : \mathbf{R}^4 \longrightarrow \mathbf{R}^4$ and imposed on his operator D an invariance condiiton of the form

$$D(\psi(g \cdot X) = g \cdot (D\psi)(X)$$

and many D have this property; Dirac's operator is one of the simplest possibilities. Here is then an example of substantial progress in physics (Dirac's equation for the electron) which is inspired by group theory.

Another significant application of Lie group theory to physics is the "eighth-fold way" theory of Gell-Mann and Ne'eman. According to mathematical quantum mechanics, if G is the group of operations leaving a physical object invariant the possible quantum states of the object are in correspondence with the representations of G. Here a representation of a Lie group G means a homomorphism of G into a unitary group $\mathbf{U}(n)$. Such representations have a well-developed structure theory which for many groups, for example $G = \mathbf{SU}(3)$, (the group of 3×3 unitary matrices of determinant one) have interpretations in physics.

An eight member structure consisting of so-called Baryons (the eight-fold way) showed similarities to the adjoint representation of $\mathbf{SU}(3)$, the representation which to a $g \in \mathbf{SU}(3)$ associates the linear transformation $X \longrightarrow gX^{-1}g$ of the space of 3×3 skew-Hermitian matrices of trace 0. But $\mathbf{SU}(3)$ has also an irreducible 10-dimensional representation π. This led to the postulation of a new baryon (Ω^-) which together with nine known baryons formed a multiplet with symmetry properties prescribed by π. Such a baryon was later discovered in a bubble chamber photograph.

Finite groups. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbf{C}, (X_i)$ a basis and c^i_{jk} the structural constants given by

$$[X_j, X_k] = \sum_i c^i_{jk} X_i.$$

A theorem of Chevalley [3] states that the basis X_i can be chosen such that the c^i_{jk} are *integers* ([6], p. 195). While the theorem is not particularly difficult to prove it has major implications. In particular, if we read the c^i_{jk} mod p we have a Lie algebra over a field of characteristic p. By considering the finite fields and the automorphism groups of the ensuing Lie algebras Chevalley constructed the simple finite groups which correspond to the Killing-Cartan classification. These groups and certain twisted analogs are called finite groups of Lie type.

Several simple finite groups did not fit into this construction. However, it became more and more difficult to construct such groups which therefore acquired the designation *sporadic.* Specialist began to suspect that there would only be finitely many of those and that all finite simple groups could therefore be explicitly constructed. This enormous project has now been completed (1981) with the following result.

Main theorem. *The simple groups of Lie type and 26 others explicitly constructed sporadic groups of order from* 7920 *to*

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71,$$

constitute all finite simple groups.

The largest of these 26 is called "The Friendly Giant" or "The Monster". The original proof is the result of a number of journal articles of total length over 10,000 pages.

In this lecture I have not tried to describe the breathtaking richness which is to be found in the modern theory of Lie groups. My purpose has been more to point out its influence on other branches of mathematics. Although I had to omit the applications in differential geometry, integral geometry, analysis, topology and combinatorics, I hope you have got an impression of the position that Lie group theory holds in modern mathematics.

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