

**1.9. The MacLane coherence theorem.** In a monoidal category, one can form  $n$ -fold tensor products of any ordered sequence of objects  $X_1, \dots, X_n$ . Namely, such a product can be attached to any parenthesizing of the expression  $X_1 \otimes \dots \otimes X_n$ , and such products are, in general, distinct objects of  $\mathcal{C}$ .

However, for  $n = 3$ , the associativity isomorphism gives a canonical identification of the two possible parenthesizings,  $(X_1 \otimes X_2) \otimes X_3$  and  $X_1 \otimes (X_2 \otimes X_3)$ . An easy combinatorial argument then shows that one can identify any two parenthesized products of  $X_1, \dots, X_n$ ,  $n \geq 3$ , using a chain of associativity isomorphisms.

We would like to say that for this reason we can completely ignore parentheses in computations in any monoidal category, identifying all possible parenthesized products with each other. But this runs into the following problem: for  $n \geq 4$  there may be two or more different chains of associativity isomorphisms connecting two different parenthesizings, and a priori it is not clear that they provide the same identification.

Luckily, for  $n = 4$ , this is settled by the pentagon axiom, which states exactly that the two possible identifications are the same. But what about  $n > 4$ ?

This problem is solved by the following theorem of MacLane, which is the first important result in the theory of monoidal categories.

**Theorem 1.9.1.** (*MacLane's Coherence Theorem*) [ML] *Let  $X_1, \dots, X_n \in \mathcal{C}$ . Let  $P_1, P_2$  be any two parenthesized products of  $X_1, \dots, X_n$  (in this order) with arbitrary insertions of unit objects  $\mathbf{1}$ . Let  $f, g : P_1 \rightarrow P_2$  be two isomorphisms, obtained by composing associativity and unit isomorphisms and their inverses possibly tensored with identity morphisms. Then  $f = g$ .*

*Proof.* We derive this theorem as a corollary of the MacLane's strictness Theorem 1.8.5. Let  $L : \mathcal{C} \rightarrow \mathcal{C}'$  be a monoidal equivalence between  $\mathcal{C}$  and a strict monoidal category  $\mathcal{C}'$ . Consider a diagram in  $\mathcal{C}$  representing  $f$  and  $g$  and apply  $L$  to it. Over each arrow of the resulting diagram representing an associativity isomorphism, let us build a rectangle as in (1.4.1), and do similarly for the unit morphisms. This way we obtain a prism one of whose faces consists of identity maps (associativity and unit isomorphisms in  $\mathcal{C}'$ ) and whose sides are commutative. Hence, the other face is commutative as well, i.e.,  $f = g$ .  $\square$

As we mentioned, this implies that any two parenthesized products of  $X_1, \dots, X_n$  with insertions of unit objects are indeed canonically isomorphic, and thus one can safely identify all of them with each other

and ignore bracketings in calculations in a monoidal category. We will do so from now on, unless confusion is possible.

**1.10. Rigid monoidal categories.** Let  $(\mathcal{C}, \otimes, \mathbf{1}, a, \iota)$  be a monoidal category, and let  $X$  be an object of  $\mathcal{C}$ . In what follows, we suppress the unit morphisms  $l, r$ .

**Definition 1.10.1.** A *right dual* of an object  $X$  in  $\mathcal{C}$  is an object  $X^*$  in  $\mathcal{C}$  equipped with morphisms  $\text{ev}_X : X^* \otimes X \rightarrow \mathbf{1}$  and  $\text{coev}_X : \mathbf{1} \rightarrow X \otimes X^*$ , called the *evaluation* and *coevaluation* morphisms, such that the compositions

$$(1.10.1) \quad X \xrightarrow{\text{coev}_X \otimes \text{Id}_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X, X^*, X}} X \otimes (X^* \otimes X) \xrightarrow{\text{Id}_X \otimes \text{ev}_X} X,$$

$$(1.10.2) \quad X^* \xrightarrow{\text{Id}_{X^*} \otimes \text{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{a_{X^*, X, X^*}^{-1}} (X^* \otimes X) \otimes X^* \xrightarrow{\text{ev}_X \otimes \text{Id}_{X^*}} X^*$$

are the identity morphisms.

**Definition 1.10.2.** A *left dual* of an object  $X$  in  $\mathcal{C}$  is an object  ${}^*X$  in  $\mathcal{C}$  equipped with morphisms  $\text{ev}'_X : X \otimes {}^*X \rightarrow \mathbf{1}$  and  $\text{coev}'_X : \mathbf{1} \rightarrow {}^*X \otimes X$  such that the compositions

$$(1.10.3) \quad X \xrightarrow{\text{Id}_X \otimes \text{coev}'_X} X \otimes ({}^*X \otimes X) \xrightarrow{a_{X, {}^*X, X}^{-1}} (X \otimes {}^*X) \otimes X \xrightarrow{\text{ev}'_X \otimes \text{Id}_X} X,$$

$$(1.10.4) \quad {}^*X \xrightarrow{\text{coev}'_X \otimes \text{Id}_X} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{a_{{}^*X, X, {}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{\text{Id}_{{}^*X} \otimes \text{ev}'_X} {}^*X$$

are the identity morphisms.

**Remark 1.10.3.** It is obvious that if  $X^*$  is a right dual of an object  $X$  then  $X$  is a left dual of  $X^*$  with  $\text{ev}'_{X^*} = \text{ev}_X$  and  $\text{coev}'_{X^*} = \text{coev}_X$ , and vice versa. Also, in any monoidal category,  $\mathbf{1}^* = {}^*\mathbf{1} = \mathbf{1}$  with the structure morphisms  $\iota$  and  $\iota^{-1}$ . Also note that changing the order of tensor product switches right duals and left duals, so to any statement about right duals there corresponds a symmetric statement about left duals.

**Proposition 1.10.4.** *If  $X \in \mathcal{C}$  has a right (respectively, left) dual object, then it is unique up to a unique isomorphism.*

*Proof.* Let  $X_1^*, X_2^*$  be two right duals to  $X$ . Denote by  $e_1, c_1, e_2, c_2$  the corresponding evaluation and coevaluation morphisms. Then we have

a morphism  $\alpha : X_1^* \rightarrow X_2^*$  defined as the composition

$$X_1^* \xrightarrow{\text{Id}_{X_1^*} \otimes c_2} X_1^* \otimes (X \otimes X_2^*) \xrightarrow{a_{X_1^*, X, X_2^*}^{-1}} (X_1^* \otimes X) \otimes X_2^* \xrightarrow{e_1 \otimes \text{Id}_{X_2^*}} X_2^*.$$

Similarly one defines a morphism  $\beta : X_2^* \rightarrow X_1^*$ . We claim that  $\beta \circ \alpha$  and  $\alpha \circ \beta$  are the identity morphisms, so  $\alpha$  is an isomorphism. Indeed consider the following diagram:

$$\begin{array}{ccccc} X_1^* & \xrightarrow{\text{Id} \otimes c_1} & X_1^* \otimes X \otimes X_1^* & & \\ \text{Id} \otimes c_2 \downarrow & & \text{Id} \otimes c_2 \otimes \text{Id} \downarrow & \searrow \text{Id} & \\ X_1^* \otimes X \otimes X_2^* & \xrightarrow{\text{Id} \otimes c_1} & X_1^* \otimes X \otimes X_2^* \otimes X \otimes X_1^* & \xrightarrow{\text{Id} \otimes e_2 \otimes \text{Id}} & X_1^* \otimes X \otimes X_1^* \\ e_1 \otimes \text{Id} \downarrow & & e_1 \otimes \text{Id} \downarrow & & \downarrow e_1 \otimes \text{Id} \\ X_2^* & \xrightarrow{\text{Id} \otimes c_1} & X_2^* \otimes X \otimes X_1^* & \xrightarrow{e_2 \otimes \text{Id}} & X_1^*. \end{array}$$

Here we suppress the associativity constraints. It is clear that the three small squares commute. The triangle in the upper right corner commutes by axiom (1.10.1) applied to  $X_2^*$ . Hence, the perimeter of the diagram commutes. The composition through the top row is the identity by (1.10.2) applied to  $X_1^*$ . The composition through the bottom row is  $\beta \circ \alpha$  and so  $\beta \circ \alpha = \text{Id}$ . The proof of  $\alpha \circ \beta = \text{Id}$  is completely similar.

Moreover, it is easy to check that  $\alpha : X_1^* \rightarrow X_2^*$  is the only isomorphism which preserves the evaluation and coevaluation morphisms. This proves the proposition for right duals. The proof for left duals is similar.  $\square$

**Exercise 1.10.5.** Fill in the details in the proof of Proposition 1.10.4.

If  $X, Y$  are objects in  $\mathcal{C}$  which have right duals  $X^*, Y^*$  and  $f : X \rightarrow Y$  is a morphism, one defines the *right dual*  $f^* : Y^* \rightarrow X^*$  of  $f$  as the composition

$$(1.10.5) \quad \begin{aligned} Y^* &\xrightarrow{\text{Id}_{Y^*} \otimes \text{coev}_X} Y^* \otimes (X \otimes X^*) \xrightarrow{a_{Y^*, X, X^*}^{-1}} (Y^* \otimes X) \otimes X^* \\ &\xrightarrow{(\text{Id}_{Y^*} \otimes f) \otimes \text{Id}_{X^*}} (Y^* \otimes Y) \otimes X^* \xrightarrow{\text{ev}_Y \otimes \text{Id}_{X^*}} X^*. \end{aligned}$$

Similarly, if  $X, Y$  are objects in  $\mathcal{C}$  which have left duals  ${}^*X, {}^*Y$  and  $f : X \rightarrow Y$  is a morphism, one defines the *left dual*  ${}^*f : {}^*Y \rightarrow {}^*X$  of  $f$  as the composition

$$(1.10.6) \quad \begin{aligned} {}^*Y &\xrightarrow{\text{coev}'_X \otimes \text{Id}_{Y^*}} ({}^*X \otimes X) \otimes {}^*Y \xrightarrow{a_{{}^*X, X, {}^*Y}} {}^*X \otimes (X \otimes {}^*Y) \\ &\xrightarrow{\text{Id}_{{}^*X} \otimes (f \otimes \text{Id}_{{}^*Y})} {}^*X \otimes (Y \otimes {}^*Y) \xrightarrow{\text{Id}_{{}^*X} \otimes \text{ev}'_Y} {}^*X. \end{aligned}$$

**Exercise 1.10.6.** Let  $\mathcal{C}, \mathcal{D}$  be monoidal categories. Suppose

$$(F, J) : \mathcal{C} \rightarrow \mathcal{D}$$

is a monoidal functor with the corresponding isomorphism  $\varphi : \mathbf{1} \rightarrow F(\mathbf{1})$ . Let  $X$  be an object in  $\mathcal{C}$  with a right dual  $X^*$ . Prove that  $F(X^*)$  is a right dual of  $F(X)$  with the evaluation and coevaluation given by

$$\begin{aligned} \text{ev}_{F(X)} &: F(X^*) \otimes F(X) \xrightarrow{J_{X, X^*}} F(X^* \otimes X) \xrightarrow{F(\text{ev}_X)} F(\mathbf{1}) = \mathbf{1}, \\ \text{coev}_{F(X)} &: \mathbf{1} = F(\mathbf{1}) \xrightarrow{F(\text{coev}_X)} F(X \otimes X^*) \xrightarrow{J_{X, X^*}^{-1}} F(X) \otimes F(X^*). \end{aligned}$$

State and prove a similar result for left duals.

**Proposition 1.10.7.** *Let  $\mathcal{C}$  be a monoidal category.*

(i) *Let  $U, V, W$  be objects in  $\mathcal{C}$  admitting right (respectively, left) duals, and let  $f : V \rightarrow W$ ,  $g : U \rightarrow V$  be morphisms in  $\mathcal{C}$ . Then  $(f \circ g)^* = g^* \circ f^*$  (respectively,  $*(f \circ g) = *g \circ *f$ ).*

(ii) *If  $U, V$  have right (respectively, left) duals then the object  $V^* \otimes U^*$  (respectively,  $*V \otimes *U$ ) has a natural structure of a right (respectively, left) dual to  $U \otimes V$ .*

**Exercise 1.10.8.** Prove Proposition 1.10.7.

**Proposition 1.10.9.** (i) *If an object  $V$  has a right dual  $V^*$  then there are natural adjunction isomorphisms*

$$(1.10.7) \quad \text{Hom}(U \otimes V, W) \xrightarrow{\sim} \text{Hom}(U, W \otimes V^*),$$

$$(1.10.8) \quad \text{Hom}(V^* \otimes U, W) \xrightarrow{\sim} \text{Hom}(U, V \otimes W).$$

*Thus, the functor  $\bullet \otimes V^*$  is right adjoint to  $\bullet \otimes V$  and  $V \otimes \bullet$  is right adjoint to  $V^* \otimes \bullet$ .*

(ii) *If an object  $V$  has a left dual  $*V$  then there are natural adjunction isomorphisms*

$$(1.10.9) \quad \text{Hom}(U \otimes *V, W) \xrightarrow{\sim} \text{Hom}(U, W \otimes V),$$

$$(1.10.10) \quad \text{Hom}(V \otimes U, W) \xrightarrow{\sim} \text{Hom}(U, *V \otimes W).$$

*Thus, the functor  $\bullet \otimes V$  is right adjoint to  $\bullet \otimes *V$  and  $*V \otimes \bullet$  is right adjoint to  $V \otimes \bullet$ .*

*Proof.* The isomorphism in (1.10.7) is given by

$$f \mapsto (f \otimes \text{Id}_{V^*}) \circ (\text{Id}_U \otimes \text{coev}_V)$$

and has the inverse

$$g \mapsto (\text{Id}_W \otimes \text{ev}_V) \circ (g \otimes \text{Id}_V).$$

The other isomorphisms are similar, and are left to the reader as an exercise. <sup>7</sup>  $\square$

**Remark 1.10.10.** Proposition 1.10.9 provides another proof of Proposition 1.10.4. Namely, setting  $U = \mathbf{1}$  and  $V = X$  in (1.10.8), we obtain a natural isomorphism  $\mathbf{Hom}(X^*, W) \cong \mathbf{Hom}(\mathbf{1}, X \otimes W)$  for any right dual  $X^*$  of  $X$ . Hence, if  $Y_1, Y_2$  are two such duals then there is a natural isomorphism  $\mathbf{Hom}(Y_1, W) \cong \mathbf{Hom}(Y_2, W)$ , whence there is a canonical isomorphism  $Y_1 \cong Y_2$  by Yoneda's Lemma. The proof for left duals is similar.

**Definition 1.10.11.** A monoidal category  $\mathcal{C}$  is called *rigid* if every object  $X \in \mathcal{C}$  has a right dual object and a left dual object.

**Example 1.10.12.** The category  $\mathbf{Vec}$  of finite dimensional  $k$ -vector spaces is rigid: the right and left dual to a finite dimensional vector space  $V$  are its dual space  $V^*$ , with the evaluation map  $\text{ev}_V : V^* \otimes V \rightarrow k$  being the contraction, and the coevaluation map  $\text{coev}_V : k \rightarrow V \otimes V^*$  being the usual embedding. On the other hand, the category  $\mathbf{Vec}$  of all  $k$ -vector spaces is not rigid, since for infinite dimensional spaces there is no coevaluation maps (indeed, suppose that  $c : k \rightarrow V \otimes Y$  is a coevaluation map, and consider the subspace  $V'$  of  $V$  spanned by the first component of  $c(1)$ ; this subspace finite dimensional, and yet the composition  $V \rightarrow V \otimes Y \otimes V \rightarrow V$ , which is supposed to be the identity map, lands in  $V'$  - a contradiction).

**Example 1.10.13.** The category  $\mathbf{Rep}(G)$  of finite dimensional  $k$ -representations of a group  $G$  is rigid: for a finite dimensional representation  $V$ , the (left or right) dual representation  $V^*$  is the usual dual space (with the evaluation and coevaluation maps as in Example 1.10.12), and with the  $G$ -action given by  $\rho_{V^*}(g) = (\rho_V(g)^{-1})^*$ . Similarly, the category  $\mathbf{Rep}(\mathfrak{g})$  of finite dimensional representations of a Lie algebra  $\mathfrak{g}$  is rigid, with  $\rho_{V^*}(a) = -\rho_V(a)^*$ .

**Example 1.10.14.** The category  $\mathbf{Vec}_G$  is rigid if and only if the monoid  $G$  is a group; namely,  $\delta_g^* = {}^*\delta_g = \delta_{g^{-1}}$  (with the obvious structure maps). More generally, for any group  $G$  and 3-cocycle  $\omega \in Z^3(G, k^\times)$ , the category  $\mathbf{Vec}_G^\omega$  is rigid. Namely, assume for simplicity that the cocycle  $\omega$  is normalized (as we know, we can do so without loss of generality). Then we can define duality as above, and normalize the coevaluation morphisms of  $\delta_g$  to be the identities. The evaluation morphisms will then be defined by the formula  $\text{ev}_{\delta_g} = \omega(g, g^{-1}, g)$ .

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<sup>7</sup>A convenient way to do computations in this and previous Propositions is using the graphical calculus (see [K, Chapter XIV]).

It follows from Proposition 1.10.4 that in a monoidal category  $\mathcal{C}$  with right (respectively, left) duals, one can define the (contravariant) *right* (respectively, *left*) *duality functor*  $\mathcal{C} \rightarrow \mathcal{C}$  by  $X \mapsto X^*$ ,  $f \mapsto f^*$  (respectively,  $X \mapsto {}^*X$ ,  $f \mapsto {}^*f$ ) for every object  $X$  and morphism  $f$  in  $\mathcal{C}$ . By Proposition 1.10.7(ii), these functors are anti-monoidal, in the sense that they define monoidal functors  $\mathcal{C}^\vee \rightarrow \mathcal{C}^{op}$ ; hence the functors  $X \rightarrow X^{**}$ ,  $X \rightarrow {}^{**}X$  are monoidal. Also, it follows from Proposition 1.10.9 that the functors of right and left duality, when they are defined, are fully faithful (it suffices to use (i) for  $U = X^*$ ,  $V = Y$ ,  $W = \mathbf{1}$ ).

Moreover, it follows from Remark 1.10.3 that in a rigid monoidal category, the functors of right and left duality are mutually quasi-inverse monoidal equivalences of categories  $\mathcal{C}^\vee \rightarrow \mathcal{C}^{op}$  (so for rigid categories, the two notions of opposite category are the same up to equivalence). This implies that the functors  $X \rightarrow X^{**}$ ,  $X \rightarrow {}^{**}X$  are mutually quasi-inverse monoidal autoequivalences. We will see later in Example 1.27.2 that these autoequivalences may be nontrivial; in particular, it is possible that objects  $V^*$  and  ${}^*V$  are not isomorphic.

**Exercise 1.10.15.** Show that if  $\mathcal{C}, \mathcal{D}$  are rigid monoidal categories,  $F_1, F_2 : \mathcal{C} \rightarrow \mathcal{D}$  are monoidal functors, and  $\eta : F_1 \rightarrow F_2$  is a morphism of monoidal functors, then  $\eta$  is an isomorphism.<sup>8</sup>

**Exercise 1.10.16.** Let  $A$  be an algebra. Show that  $M \in A - \mathbf{bimod}$  has a left (respectively, right) dual if and only if it is finitely generated projective when considered as a left (respectively, right)  $A$ -module. Similarly, if  $A$  is commutative,  $M \in A - \mathbf{mod}$  has a left and right dual if and only if it is finitely generated projective.

**1.11. Invertible objects.** Let  $\mathcal{C}$  be a rigid monoidal category.

**Definition 1.11.1.** An object  $X$  in  $\mathcal{C}$  is *invertible* if  $\text{ev}_X : X^* \otimes X \rightarrow \mathbf{1}$  and  $\text{coev}_X : \mathbf{1} \rightarrow X \otimes X^*$  are isomorphisms.

Clearly, this notion categorifies the notion of an invertible element in a monoid.

**Example 1.11.2.** The objects  $\delta_g$  in  $\text{Vec}_G^\omega$  are invertible.

**Proposition 1.11.3.** *Let  $X$  be an invertible object in  $\mathcal{C}$ . Then*

- (i)  ${}^*X \cong X^*$  and  $X^*$  is invertible;
- (ii) if  $Y$  is another invertible object then  $X \otimes Y$  is invertible.

*Proof.* Dualizing  $\text{coev}_X$  and  $\text{ev}_X$  we get isomorphisms  $X \otimes {}^*X \cong \mathbf{1}$  and  ${}^*X \otimes X \cong \mathbf{1}$ . Hence  ${}^*X \cong {}^*X \otimes X \otimes X^* \cong X^*$ . In any rigid category the evaluation and coevaluation morphisms for  ${}^*X$  can be

<sup>8</sup>As we have seen in Remark 1.6.6, this is false for non-rigid categories.

defined by  $\text{ev}_{*X} := {}^*\text{coev}_X$  and  $\text{coev}_{*X} := {}^*\text{ev}_X$ , so  $*X$  is invertible. The second statement follows from the fact that  $\text{ev}_{X \otimes Y}$  can be defined as a composition of  $\text{ev}_X$  and  $\text{ev}_Y$  and similarly  $\text{coev}_{X \otimes Y}$  can be defined as a composition of  $\text{coev}_Y$  and  $\text{coev}_X$ .  $\square$

Proposition 1.11.3 implies that invertible objects of  $\mathcal{C}$  form a monoidal subcategory  $\text{Inv}(\mathcal{C})$  of  $\mathcal{C}$ .

**Example 1.11.4. Gr-categories.** Let us classify rigid monoidal categories  $\mathcal{C}$  where all objects are invertible and all morphisms are isomorphisms. We may assume that  $\mathcal{C}$  is skeletal, i.e. there is only one object in each isomorphism class, and objects form a group  $G$ . Also, by Proposition 1.2.7,  $\text{End}(\mathbf{1})$  is an abelian group; let us denote it by  $A$ . Then for any  $g \in G$  we can identify  $\text{End}(g)$  with  $A$ , by sending  $f \in \text{End}(g)$  to  $f \otimes \text{Id}_{g^{-1}} \in \text{End}(\mathbf{1}) = A$ . Then we have an action of  $G$  on  $A$  by

$$a \in \text{End}(\mathbf{1}) \mapsto g(a) := \text{Id}_g \otimes a \in \text{End}(g).$$

Let us now consider the associativity isomorphism. It is defined by a function  $\omega : G \times G \times G \rightarrow A$ . The pentagon relation gives

$$(1.11.1)$$

$$\omega(g_1 g_2, g_3, g_4) \omega(g_1, g_2, g_3 g_4) = \omega(g_1, g_2, g_3) \omega(g_1, g_2 g_3, g_4) g_1(\omega(g_2, g_3, g_4)),$$

for all  $g_1, g_2, g_3, g_4 \in G$ , which means that  $\omega$  is a 3-cocycle of  $G$  with coefficients in the (generally, nontrivial)  $G$ -module  $A$ . We see that any such 3-cocycle defines a rigid monoidal category, which we will call  $\mathcal{C}_G^\omega(A)$ . The analysis of monoidal equivalences between such categories is similar to the case when  $A$  is a trivial  $G$ -module, and yields that for a given group  $G$  and  $G$ -module  $A$ , equivalence classes of  $\mathcal{C}_G^\omega$  are parametrized by  $H^3(G, A)/\text{Out}(G)$ .

Categories of the form  $\mathcal{C}_G^\omega(A)$  are called Gr-categories, and were studied in [Si].

**1.12. Tensor and multitensor categories.** Now we will start considering monoidal structures on abelian categories. For the sake of brevity, we will not recall the basic theory of abelian categories; let us just recall the Freyd-Mitchell theorem stating that abelian categories can be characterized as full subcategories of categories of left modules over rings, which are closed under taking direct sums, as well as kernels, cokernels, and images of morphisms. This allows one to visualize the main concepts of the theory of abelian categories in terms of the classical theory of modules over rings.

Recall that an abelian category  $\mathcal{C}$  is said to be  $k$ -linear (or defined over  $k$ ) if for any  $X, Y$  in  $\mathcal{C}$ ,  $\text{Hom}(X, Y)$  is a  $k$ -vector space, and composition of morphisms is bilinear.

**Definition 1.12.1.** A  $k$ -linear abelian category is said to be *locally finite* if it is essentially small<sup>9</sup>, and the following two conditions are satisfied:

- (i) for any two objects  $X, Y$  in  $\mathcal{C}$ , the space  $\mathbf{Hom}(X, Y)$  is finite dimensional;
- (ii) every object in  $\mathcal{C}$  has finite length.

Almost all abelian categories we will consider will be locally finite.

**Proposition 1.12.2.** *In a locally finite abelian category  $\mathcal{C}$ ,  $\mathbf{Hom}(X, Y) = 0$  if  $X, Y$  are simple and non-isomorphic, and  $\mathbf{Hom}(X, X) = k$  for any simple object  $X$ .*

*Proof.* Recall Schur's lemma: if  $X, Y$  are simple objects of an abelian category, and  $f \in \mathbf{Hom}(X, Y)$ , then  $f = 0$  or  $f$  is an isomorphism. This implies that  $\mathbf{Hom}(X, Y) = 0$  if  $X, Y$  are simple and non-isomorphic, and  $\mathbf{Hom}(X, X)$  is a division algebra; since  $k$  is algebraically closed, condition (i) implies that  $\mathbf{Hom}(X, X) = k$  for any simple object  $X \in \mathcal{C}$ .  $\square$

Also, the Jordan-Hölder and Krull-Schmidt theorems hold in any locally finite abelian category  $\mathcal{C}$ .

**Definition 1.12.3.** Let  $\mathcal{C}$  be a locally finite  $k$ -linear abelian rigid monoidal category. We will call  $\mathcal{C}$  a *multitensor category* over  $k$  if the bifunctor  $\otimes$  is bilinear on morphisms. If in addition  $\mathbf{End}(\mathbf{1}) \cong k$  then we will call  $\mathcal{C}$  a *tensor category*.

A *multifusion category* is a semisimple multitensor category with finitely many isomorphism simple objects. A *fusion category* is a semisimple tensor category with finitely many isomorphism simple objects.

**Example 1.12.4.** The categories  $\mathbf{Vec}$  of finite dimensional  $k$ -vector spaces,  $\mathbf{Rep}(G)$  of finite dimensional  $k$ -representations of a group  $G$  (or algebraic representations of an affine algebraic group  $G$ ),  $\mathbf{Rep}(\mathfrak{g})$  of finite dimensional representations of a Lie algebra  $\mathfrak{g}$ , and  $\mathbf{Vec}_G^\omega$  of  $G$ -graded finite dimensional  $k$ -vector spaces with associativity defined by a 3-cocycle  $\omega$  are tensor categories. If  $G$  is a finite group,  $\mathbf{Rep}(G)$  is a fusion category. In particular,  $\mathbf{Vec}$  is a fusion category.

**Example 1.12.5.** Let  $A$  be a finite dimensional semisimple algebra over  $k$ . Let  $A$ -bimod be the category of finite dimensional  $A$ -bimodules with bimodule tensor product over  $A$ , i.e.,

$$(M, N) \mapsto M \otimes_A N.$$

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<sup>9</sup>Recall that a category is called essentially small if its isomorphism classes of objects form a set.

Then  $\mathcal{C}$  is a multitensor category with the unit object  $\mathbf{1} = A$ , the left dual defined by  $M \mapsto \mathbf{Hom}({}_A M, {}_A A)$ , and the right dual defined by  $M \mapsto \mathbf{Hom}(M_A, A_A)$ .<sup>10</sup> The category  $\mathcal{C}$  is tensor if and only if  $A$  is simple, in which case it is equivalent to  $k\text{-Vec}$ . More generally, if  $A$  has  $n$  matrix blocks, the category  $\mathcal{C}$  can be alternatively described as the category whose objects are  $n$ -by- $n$  matrices of vector spaces,  $V = (V_{ij})$ , and the tensor product is matrix multiplication:

$$(V \otimes W)_{il} = \bigoplus_{j=1}^n V_{ij} \otimes W_{jl}.$$

This category will be denoted by  $M_n(\text{Vec})$ . It is a multifusion category.

In a similar way, one can define the multitensor category  $M_n(\mathcal{C})$  of  $n$ -by- $n$  matrices of objects of a given multitensor category  $\mathcal{C}$ . If  $\mathcal{C}$  is a multifusion category, so is  $M_n(\mathcal{C})$ .

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<sup>10</sup>Note that if  $A$  is a finite dimensional non-semisimple algebra then the category of finite dimensional  $A$ -bimodules is not rigid, since the duality functors defined as above do not satisfy rigidity axioms (cf. Exercise 1.10.16).

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Spring 2009

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