1.13. **Exactness of the tensor product.**

Proposition 1.13.1. *(see* [BaKi, 2.1.8]*) Let $\mathcal{C}$ be a multitensor category. Then the bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is exact in both factors (i.e.,* biexact*).*

Proof. The proposition follows from the fact that by Proposition 1.10.9, the functors $V\otimes$ and $\otimes V$ have left and right adjoint functors (the functors of tensoring with the corresponding duals), and any functor between abelian categories which has a left and a right adjoint functor is exact. □

Remark 1.13.2. The proof of Proposition 1.13.1 shows that the biadditivity of the functor $\otimes$ holds automatically in any rigid monoidal abelian category. However, this is not the case for bilinearity of $\otimes$, and thus condition of bilinearity of tensor product in the definition of a multitensor category is not redundant.

This may be illustrated by the following example. Let $\mathcal{C}$ be the category of finite dimensional $\mathbb{C}$-bimodules in which the left and right actions of $\mathbb{R}$ coincide. This category is $\mathbb{C}$-linear abelian; namely, it is semisimple with two simple objects $\mathbb{C}_+ = \mathbf{1}$ and $\mathbb{C}_-$, both equal to $\mathbb{C}$ as a real vector space, with bimodule structures $(a, b)z = azb$ and $(a, b)z = az\bar{b}$, respectively. It is also also rigid monoidal, with $\otimes$ being the tensor product of bimodules. But the tensor product functor is not $\mathbb{C}$-bilinear on morphisms (it is only $\mathbb{R}$-bilinear).

Definition 1.13.3. A *multiring category* over k is a locally finite k-linear abelian monoidal category $\mathcal{C}$ with biexact tensor product. If in addition $\mathsf{End}(\mathbf{1}) = k$, we will call $\mathcal{C}$ a *ring category*.

Thus, the difference between this definition and the definition of a (multi)tensor category is that we don't require the existence of duals, but instead require the biexactness of the tensor product. Note that Proposition 1.13.1 implies that any multitensor category is a multiring category, and any tensor category is a ring category.

Corollary 1.13.4. *For any pair of morphisms f_1, f_2 in a multiring category $\mathcal{C}$ one has* $\mathrm{Im}(f_1 \otimes f_2) = \mathrm{Im}(f_1) \otimes \mathrm{Im}(f_2)$.

Proof. Let I_1, I_2 be the images of f_1, f_2. Then the morphisms $f_i : X_i \to Y_i$, $i = 1, 2$, have decompositions $X_i \to I_i \to Y_i$, where the sequences

$$X_i \to I_i \to 0, \; 0 \to I_i \to Y_i$$

are exact. Tensoring the sequence $X_1 \to I_1 \to 0$ with I_2, by Proposition 1.13.1, we get the exact sequence

$$X_1 \otimes I_2 \to I_1 \otimes I_2 \to 0$$

Tenosring X_1 with the sequence $X_2 \to I_2 \to 0$, we get the exact sequence

$$X_1 \otimes X_2 \to X_1 \otimes I_2 \to 0.$$

Combining these, we get an exact sequence

$$X_1 \otimes X_2 \to I_1 \otimes I_2 \to 0.$$

Arguing similarly, we show that the sequence

$$0 \to I_1 \otimes I_2 \to Y_1 \otimes Y_2$$

is exact. This implies the statement. □

Proposition 1.13.5. *If $\mathcal{C}$ is a multiring category with right duals, then the right dualization functor is exact. The same applies to left duals.*

Proof. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence. We need to show that the sequence $0 \to Z^* \to Y^* \to X^* \to 0$ is exact. Let T be any object of $\mathcal{C}$, and consider the sequence

$$0 \to \mathsf{Hom}(T, Z^*) \to \mathsf{Hom}(T, Y^*) \to \mathsf{Hom}(T, X^*).$$

By Proposition 1.10.9, it can be written as

$$0 \to \mathsf{Hom}(T \otimes Z, \mathbf{1}) \to \mathsf{Hom}(T \otimes Y, \mathbf{1}) \to \mathsf{Hom}(T \otimes X, \mathbf{1}),$$

which is exact, since the sequence

$$T \otimes X \to T \otimes Y \to T \otimes Z \to 0$$

is exact, by the exactness of the functor $T\otimes$. This implies that the sequence $0 \to Z^* \to Y^* \to X^*$ is exact.

Similarly, consider the sequence

$$0 \to \mathsf{Hom}(X^*, T) \to \mathsf{Hom}(Y^*, T) \to \mathsf{Hom}(Z^*, T).$$

By Proposition 1.10.9, it can be written as

$$0 \to \mathsf{Hom}(\mathbf{1}, X \otimes T) \to \mathsf{Hom}(\mathbf{1}, Y \otimes T) \to \mathsf{Hom}(\mathbf{1}, Z \otimes T),$$

which is exact since the sequence

$$0 \to X \otimes T \to Y \otimes T \to Z \otimes T$$

is exact, by the exactness of the functor $\otimes T$. This implies that the sequence $Z^* \to Y^* \to X^* \to 0$ is exact. □

Proposition 1.13.6. *Let P be a projective object in a multiring category $\mathcal{C}$. If $X \in \mathcal{C}$ has a right dual, then the object $P \otimes X$ is projective. Similarly, if $X \in \mathcal{C}$ has a left dual, then the object $X \otimes P$ is projective.*

Proof. In the first case by Proposition 1.10.9 we have $\mathsf{Hom}(P \otimes X, Y) = \mathsf{Hom}(P, Y \otimes X^*)$, which is an exact functor of Y, since the functors $\otimes X^*$ and $\mathsf{Hom}(P, \bullet)$ are exact. So $P \otimes X$ is projective. The second case is similar. □

Corollary 1.13.7. *If $\mathcal{C}$ multiring category with right duals, then $\mathbf{1} \in \mathcal{C}$ is a projective object if and only if $\mathcal{C}$ is semisimple.*

Proof. If $\mathbf{1}$ is projective then for any $X \in \mathcal{C}$, $X \cong \mathbf{1} \otimes X$ is projective. This implies that $\mathcal{C}$ is semisimple. The converse is obvious. □

1.14. **Quasi-tensor and tensor functors.**

Definition 1.14.1. Let $\mathcal{C}$, $\mathcal{D}$ be multiring categories over k, and $F : \mathcal{C} \to \mathcal{D}$ be an exact and faithful functor.

(i) F is said to be a quasi-tensor functor if it is equipped with a functorial isomorphism $J : F(\bullet) \otimes F(\bullet) \to F(\bullet \otimes \bullet)$, and $F(\mathbf{1}) = \mathbf{1}$.

(ii) A quasi-tensor functor (F, J) is said to be a tensor functor if J is a monoidal structure (i.e., satisfies the monoidal structure axiom).

Example 1.14.2. The functors of Examples 1.6.1,1.6.2 and Subsection 1.7 (for the categories Vec_G^{ω}) are tensor functors. The identity functor $\mathrm{Vec}_G^{\omega_1} \to Vec_G^{\omega_2}$ for non-cohomologous 3-cocycles ω_1, ω_2 is not a tensor functor, but it can be made quasi-tensor by any choice of J.

1.15. **Semisimplicity of the unit object.**

Theorem 1.15.1. *In any multiring category, $\mathsf{End}(\mathbf{1})$ is a semisimple algebra, so it is isomorphic to a direct sum of finitely many copies of k.*

Proof. By Proposition 1.2.7, $\mathsf{End}(\mathbf{1})$ is a commutative algebra, so it is sufficient to show that for any $a \in \mathsf{End}(\mathbf{1})$ such that $a^2 = 0$ we have $a = 0$. Let $J = \mathrm{Im}(a)$. Then by Corollary 1.13.4 $J \otimes J = \mathrm{Im}(a \otimes a) = \mathrm{Im}(a^2 \otimes 1) = 0$.

Now let $K = \mathrm{Ker}(a)$. Then by Corollary 1.13.4, $K \otimes J$ is the image of $1 \otimes a$ on $K \otimes \mathbf{1}$. But since $K \otimes \mathbf{1}$ is a subobject of $\mathbf{1} \otimes \mathbf{1}$, this is the same as the image of $a \otimes 1$ on $K \otimes \mathbf{1}$, which is zero. So $K \otimes J = 0$.

Now tensoring the exact sequence $0 \to K \to \mathbf{1} \to J \to 0$ with J, and applying Proposition 1.13.1, we get that $J = 0$, so $a = 0$. □

Let $\{p_i\}_{i \in I}$ be the primitive idempotents of the algebra $\mathsf{End}(\mathbf{1})$. Let $\mathbf{1}_i$ be the image of p_i. Then we have $\mathbf{1} = \oplus_{i \in I} \mathbf{1}_i$.

Corollary 1.15.2. *In any multiring category $\mathcal{C}$ the unit object $\mathbf{1}$ is isomorphic to a direct sum of pairwise non-isomorphic indecomposable objects: $\mathbf{1} \cong \oplus_i \mathbf{1}_i$.*

Exercise 1.15.3. One has $\mathbf{1}_i \otimes \mathbf{1}_j = 0$ for $i \neq j$. There are canonical isomorphisms $\mathbf{1}_i \otimes \mathbf{1}_i \cong \mathbf{1}_i$, and $\mathbf{1}_i \cong \mathbf{1}_i{}^*$.

Let $\mathcal{C}_{ij} := \mathbf{1}_i \otimes \mathcal{C} \otimes \mathbf{1}_j$.

Definition 1.15.4. The subcategories $\mathcal{C}_{ij}$ will be called the *component* subcategories of $\mathcal{C}$.

Proposition 1.15.5. *Let $\mathcal{C}$ be a multiring category.*

(1) $\mathcal{C} = \oplus_{i,j\in I}\mathcal{C}_{ij}$. *Thus every indecomposable object of $\mathcal{C}$ belongs to some $\mathcal{C}_{ij}$.*
(2) *The tensor product maps $\mathcal{C}_{ij} \times \mathcal{C}_{kl}$ to $\mathcal{C}_{il}$, and it is zero unless $j = k$.*
(3) *The categories $\mathcal{C}_{ii}$ are ring categories with unit objects $\mathbf{1}_i$ (which are tensor categories if $\mathcal{C}$ is rigid).*
(3) *The functors of left and right duals, if they are defined, map $\mathcal{C}_{ij}$ to $\mathcal{C}_{ji}$.*

Exercise 1.15.6. Prove Proposition 1.15.5.

Proposition 1.15.5 motivates the terms "multiring category" and "multitensor category", as such a category gives us multiple ring categories, respectively tensor categories $\mathcal{C}_{ii}$.

Remark 1.15.7. Thus, a multiring category may be considered as a 2-category with objects being elements of I, 1-morphisms from j to i forming the category $\mathcal{C}_{ij}$, and 2-morphisms being 1-morphisms in $\mathcal{C}$.

Theorem 1.15.8. *(i) In a ring category with right duals, the unit object $\mathbf{1}$ is simple.*

(ii) In a multiring category with right duals, the unit object $\mathbf{1}$ is semisimple, and is a direct sum of pairwise non-isomorphic simple objects $\mathbf{1}_i$.

Proof. Clearly, (i) implies (ii) (by applying (i) to the component categories $\mathcal{C}_{ii}$). So it is enough to prove (i).

Let X be a simple subobject of $\mathbf{1}$ (it exists, since $\mathbf{1}$ has finite length). Let

$$0 \longrightarrow X \longrightarrow \mathbf{1} \longrightarrow Y \longrightarrow 0 \tag{1.15.1}$$

be the corresponding exact sequence. By Proposition 1.13.5, the right dualization functor is exact, so we get an exact sequence

$$0 \longrightarrow Y^* \longrightarrow \mathbf{1} \longrightarrow X^* \longrightarrow 0. \tag{1.15.2}$$

Tensoring this sequence with X on the left, we obtain

$$0 \longrightarrow X \otimes Y^* \longrightarrow X \longrightarrow X \otimes X^* \longrightarrow 0, \tag{1.15.3}$$

Since X is simple and $X \otimes X^* \neq 0$ (because the coevaluation morphism is nonzero) we obtain that $X \otimes X^* \cong X$. So we have a surjective composition morphism $\mathbf{1} \to X \otimes X^* \to X$. From this and (1.15.1) we have a nonzero composition morphism $\mathbf{1} \twoheadrightarrow X \hookrightarrow \mathbf{1}$. Since $\mathsf{End}(\mathbf{1}) = k$, this morphism is a nonzero scalar, whence $X = \mathbf{1}$. □

Corollary 1.15.9. *In a ring category with right duals, the evaluation morphisms are surjective and the coevaluation morphisms are injective.*

Exercise 1.15.10. Let $\mathcal{C}$ be a multiring category with right duals. and $X \in \mathcal{C}_{ij}$ and $Y \in \mathcal{C}_{jk}$ be nonzero.

(a) Show that $X \otimes Y \neq 0$.
(b) Deduce that $\text{length}(X \otimes Y) \geq \text{length}(X)\text{length}(Y)$.
(c) Show that if $\mathcal{C}$ is a ring category with right duals then an invertible object in $\mathcal{C}$ is simple.
(d) Let X be an object in a multiring category with right duals such that $X \otimes X^* \cong \mathbf{1}$. Show that X is invertible.

Example 1.15.11. An example of a ring category where the unit object is not simple is the category $\mathcal{C}$ of finite dimensional representations of the quiver of type A_2. Such representations are triples (V, W, A), where V, W are finite dimensional vector spaces, and $A : V \to W$ is a linear operator. The tensor product on such triples is defined by the formula

$$(V, W, A) \otimes (V', W', A') = (V \otimes V', W \otimes W', A \otimes A'),$$

with obvious associativity isomorphisms, and the unit object (k, k, Id). Of course, this category has neither right nor left duals.

1.16. **Grothendieck rings.** Let $\mathcal{C}$ be a locally finite abelian category over k. If X and Y are objects in $\mathcal{C}$ such that Y is simple then we denote by $[X : Y]$ the multiplicity of Y in the Jordan-Hölder composition series of X.

Recall that the Grothendieck group $\mathsf{Gr}(\mathcal{C})$ is the free abelian group generated by isomorphism classes X_i, $i \in I$ of simple objects in $\mathcal{C}$, and that to every object X in $\mathcal{C}$ we can canonically associate its class $[X] \in \mathsf{Gr}(\mathcal{C})$ given by the formula $[X] = \sum_i [X : X_i] X_i$. It is obvious that if

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

then $[Y] = [X] + [Z]$. When no confusion is possible, we will write X instead of $[X]$.

Now let $\mathcal{C}$ be a multiring category. The tensor product on $\mathcal{C}$ induces a natural multiplication on $\mathsf{Gr}(\mathcal{C})$ defined by the formula

$$X_i X_j := [X_i \otimes X_j] = \sum_{k \in I} [X_i \otimes X_j : X_k] X_k.$$

Lemma 1.16.1. *The above multiplication on* $\mathsf{Gr}(\mathcal{C})$ *is associative.*

Proof. Since the tensor product functor is exact,

$$[(X_i \otimes X_j) \otimes X_p : X_l] = \sum_k [X_i \otimes X_j : X_k][X_k \otimes X_p : X_l].$$

On the other hand,

$$[X_i \otimes (X_j \otimes X_p) : X_l] = \sum_k [X_j \otimes X_p : X_k][X_i \otimes X_k : X_l].$$

Thus the associativity of the multiplication follows from the isomorphism $(X_i \otimes X_j) \otimes X_p \cong X_i \otimes (X_j \otimes X_p)$. □

Thus $\mathsf{Gr}(\mathcal{C})$ is an associative ring with the unit $\mathbf{1}$. It is called the *Grothendieck ring* of $\mathcal{C}$.

The following proposition is obvious.

Proposition 1.16.2. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be multiring categories and* $F : \mathcal{C} \to \mathcal{D}$ *be a quasi-tensor functor. Then* F *defines a homomorphism of unital rings* $[F] : \mathsf{Gr}(\mathcal{C}) \to \mathsf{Gr}(\mathcal{D})$.

Thus, we see that (multi)ring categories categorify rings (which justifies the terminology), while quasi-tensor (in particular, tensor) functors between them categorify unital ring homomorphisms. Note that Proposition 1.15.5 may be regarded as a categorical analog of the Peirce decomposition in classical algebra.

1.17. **Groupoids.** The most basic examples of multitensor categories arise from finite groupoids. Recall that a *groupoid* is a small category where all morphisms are isomorphisms. Thus a groupoid $\mathcal{G}$ entails a set X of objects of $\mathcal{G}$ and a set G of morphisms of $\mathcal{G}$, the source and target maps $s, t : G \to X$, the composition map $\mu : G \times_X G \to G$ (where the fibered product is defined using s in the first factor and using t in the second factor), the unit morphism map $u : X \to G$, and the inversion map $i : G \to G$ satisfying certain natural axioms, see e.g. [Ren] for more details.

Here are some examples of groupoids.

(1) Any group G is a groupoid $\mathcal{G}$ with a single object whose set of morphisms to itself is G.

(2) Let X be a set and let $G = X \times X$. Then the *product groupoid* $\mathcal{G}(X) := (X, G)$ is a groupoid in which s is the first projection, t is the second projection, u is the diagonal map, and i is the permutation of factors. In this groupoid for any $x, y \in X$ there is a unique morphism from x to y.

(3) A more interesting example is the *transformation groupoid* $T(G, X)$ arising from the action of a group G on a set X. The set of objects of $T(G, X)$ is X, and arrows correspond to triples (g, x, y) where $y = gx$ with an obvious composition law. In other words, the set of morphisms is $G \times X$ and $s(g, x) = x$, $t(g, x) = gx$, $u(x) = (1, x)$, $i(g, x) = (g^{-1}, gx)$.

Let $\mathcal{G} = (X, G, \mu, s, t, u, i)$ be a finite groupoid (i.e., G is finite) and let $\mathcal{C}(\mathcal{G})$ be the category of finite dimensional vector spaces graded by the set G of morphisms of $\mathcal{G}$, i.e., vector spaces of the form $V = \oplus_{g \in G} V_g$. Introduce a tensor product on $\mathcal{C}(\mathcal{G})$ by the formula

$$(V \otimes W)_g = \bigoplus_{(g_1, g_2): g_1 g_2 = g} V_{g_1} \otimes W_{g_2}. \tag{1.17.1}$$

Then $\mathcal{C}(\mathcal{G})$ is a multitensor category. The unit object is $\mathbf{1} = \oplus_{x \in X} \mathbf{1}_x$, where $\mathbf{1}_x$ is a 1-dimensional vector space which sits in degree id_x in G. The left and right duals are defined by $(V^*)_g = ({}^*V)_g = V_{g^{-1}}$.

We invite the reader to check that the component subcategories $\mathcal{C}(\mathcal{G})_{xy}$ are the categories of vector spaces graded by $\mathsf{Mor}(y, x)$.

We see that $\mathcal{C}(\mathcal{G})$ is a tensor category if and only if $\mathcal{G}$ is a group, which is the case of Vec_G already considered in Example 1.3.6. Note also that if $X = \{1, ..., n\}$ then $\mathcal{C}(\mathcal{G}(X))$ is naturally equivalent to $M_n(\mathrm{Vec})$.

Exercise 1.17.1. Let C_i be isomorphism classes of objects in a finite groupoid $\mathcal{G}$, $n_i = |C_i|$, $x_i \in C_i$ be representatives of C_i, and $G_i = \mathrm{Aut}(x_i)$ be the corresponding automorphism groups. Show that $\mathcal{C}(\mathcal{G})$ is (non-canonically) monoidally equivalent to $\oplus_i M_{n_i}(\mathrm{Vec}_{G_i})$.

Remark 1.17.2. The finite length condition in Definition 1.12.3 is not superfluous: there exists a rigid monoidal k-linear abelian category with bilinear tensor product which contains objects of infinite length. An example of such a category is the category $\mathcal{C}$ of Jacobi matrices of finite dimensional vector spaces. Namely, the objects of $\mathcal{C}$ are semi-infinite matrices $V = \{V_{ij}\}_{ij \in \mathbb{Z}_+}$ of finite dimensional vector spaces V_{ij} with finitely many non-zero diagonals, and morphisms are matrices of linear maps. The tensor product in this category is defined by the formula

$$(V \otimes W)_{il} = \sum_j V_{ij} \otimes W_{jl}, \tag{1.17.2}$$

and the unit object $\mathbf{1}$ is defined by the condition $\mathbf{1}_{ij} = k^{\delta_{ij}}$. The left and right duality functors coincide and are given by the formula

$$(V^*)_{ij} = (V_{ji})^*. \tag{1.17.3}$$

The evaluation map is the direct sum of the canonical maps $V_{ij}^* \otimes V_{ij} \to \mathbf{1}_{jj}$, and the coevaluation map is a direct sum of the canonical maps $\mathbf{1}_{ii} \to V_{ij} \otimes V_{ij}^*$.

Note that the category $\mathcal{C}$ is a subcategory of the category $\mathcal{C}'$ of $\mathcal{G}(\mathbb{Z}_+)$-graded vector spaces with finite dimensional homogeneous components. Note also that the category $\mathcal{C}'$ is not closed under the tensor product defined by (1.17.2) but the category $\mathcal{C}$ is.

Exercise 1.17.3.

(1) Show that if X is a finite set then the group of invertible objects of the category $\mathcal{C}(\mathcal{G}(X))$ is isomorphic to $\mathsf{Aut}(X)$.

(2) Let $\mathcal{C}$ be the category of Jacobi matrices of vector spaces from Example 1.17.2. Show that the statement Exercise 1.15.10(d) fails for $\mathcal{C}$. Thus the finite length condition is important in Exercise 1.15.10.

1.18. Finite abelian categories and exact faithful functors.

Definition 1.18.1. A k-linear abelian category $\mathcal{C}$ is said to be *finite* if it is equivalent to the category $A-$mod of finite dimensional modules over a finite dimensional k-algebra A.

Of course, the algebra A is not canonically attached to the category $\mathcal{C}$; rather, $\mathcal{C}$ determines the Morita equivalence class of A. For this reason, it is often better to use the following "intrinsic" definition, which is well known to be equivalent to Definition 1.18.1:

Definition 1.18.2. A k-linear abelian category $\mathcal{C}$ is *finite* if

(i) $\mathcal{C}$ has finite dimensional spaces of morphisms;

(ii) every object of $\mathcal{C}$ has finite length;

(iii) $\mathcal{C}$ has enough projectives, i.e., every simple object of $\mathcal{C}$ has a projective cover; and

(iv) there are finitely many isomorphism classes of simple objects.

Note that the first two conditions are the requirement that $\mathcal{C}$ be locally finite.

Indeed, it is clear that if A is a finite dimensional algebra then $A-$mod clearly satisfies (i)-(iv), and conversely, if $\mathcal{C}$ satisfies (i)-(iv), then one can take $A = \mathsf{End}(P)^{op}$, where P is a projective generator of $\mathcal{C}$ (e.g., $P = \oplus_{i=1}^n P_i$, where P_i are projective covers of all the simple objects X_i).

A projective generator P of $\mathcal{C}$ represents a functor $F = F_P : \mathcal{C} \to \mathrm{Vec}$ from $\mathcal{C}$ to the category of finite dimensional k-vector spaces, given by the formula $F(X) = \mathsf{Hom}(P, X)$. The condition that P is projective translates into the exactness property of F, and the condition that P is a generator (i.e., covers any simple object) translates into the property that F is faithful (does not kill nonzero objects or morphisms). Moreover, the algebra $A = \mathsf{End}(P)^{op}$ can be alternatively defined as $\mathsf{End}(F)$, the algebra of functorial endomorphisms of F. Conversely, it is well known (and easy to show) that any exact faithful functor $F : \mathcal{C} \to \mathrm{Vec}$ is represented by a unique (up to a unique isomorphism) projective generator P.

Now let $\mathcal{C}$ be a finite k-linear abelian category, and $F_1,\ F_2 : \mathcal{C} \to \mathrm{Vec}$ be two exact faithful functors. Define the functor $F_1 \otimes F_2 : \mathcal{C} \times \mathcal{C} \to \mathrm{Vec}$ by $(F_1 \otimes F_2)(X,\, Y) := F_1(X) \otimes F_2(Y)$.

Proposition 1.18.3. *There is a canonical algebra isomorphism* α_{F_1,F_2} : $\mathsf{End}(F_1) \otimes \mathsf{End}(F_2) \cong \mathsf{End}(F_1 \otimes F_2)$ *given by*

$$\alpha_{F_1,F_2}(\eta_1 \otimes \eta_2)|_{F_1(X)\otimes F_2(Y)} := \eta_1|_{F_1(X)} \otimes \eta_2|_{F_2(Y)},$$

where $\eta_i \in \mathsf{End}(F_i)$, $i = 1, 2$.

Exercise 1.18.4. Prove Proposition 1.18.3.

1.19. **Fiber functors.** Let $\mathcal{C}$ be a k-linear abelian monoidal category.

Definition 1.19.1. A *quasi-fiber functor* on $\mathcal{C}$ is an exact faithful functor $F : \mathcal{C} \to \mathrm{Vec}$ from $\mathcal{C}$ to the category of finite dimensional k-vector spaces, such that $F(\mathbf{1}) = k$, equipped with an isomorphism $J : F(\bullet) \otimes F(\bullet) \to F(\bullet \otimes \bullet)$. If in addition J is a monoidal structure (i.e. satisfies the monoidal structure axiom), one says that F is a *fiber functor.*

Example 1.19.2. The forgetful functors $\mathrm{Vec}_G \to \mathrm{Vec}$, $\mathrm{Rep}(G) \to \mathrm{Vec}$ are naturally fiber functors, while the forgetful functor $\mathrm{Vec}_G^\omega \to \mathrm{Vec}$ is quasi-fiber, for any choice of the isomorphism J (we have seen that if ω is cohomologically nontrivial, then Vec_G^ω does not admit a fiber functor). Also, the functor $\mathrm{Loc}(X) \to \mathrm{Vec}$ on the category of local systems on a connected topological space X which attaches to a local system E its fiber E_x at a point $x \in X$ is a fiber functor, which justifies the terminology. (Note that if X is Hausdorff, then this functor can be identified with the abovementioned forgetful functor $\mathrm{Rep}(\pi_1(X, x)) \to \mathrm{Vec}$).

Exercise 1.19.3. Show that if an abelian monoidal category $\mathcal{C}$ admits a quasi-fiber functor, then it is a ring category, in which the object $\mathbf{1}$ is simple. So if in addition $\mathcal{C}$ is rigid, then it is a tensor category.

1.20. Coalgebras.

Definition 1.20.1. A *coalgebra* (with counit) over a field k is a k-vector space C together with a comultiplicaton (or coproduct) $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to k$ such that

(i) Δ is coassociative, i.e.,

$$(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta$$

as maps $C \to C^{\otimes 3}$;

(ii) one has

$$(\varepsilon \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \varepsilon) \circ \Delta = \mathrm{Id}$$

as maps $C \to C$ (the "counit axiom").

Definition 1.20.2. A left comodule over a coalgebra C is a vector space M together with a linear map $\pi : M \to C \otimes M$ (called the coaction map), such that for any $m \in M$, one has

$$(\Delta \otimes \mathrm{Id})(\pi(m)) = (\mathrm{Id} \otimes \pi)(\pi(m)),\ (\varepsilon \otimes \mathrm{Id})(\pi(m)) = m.$$

Similarly, a right comodule over C is a vector space M together with a linear map $\pi : M \to M \otimes C$, such that for any $m \in M$, one has

$$(\pi \otimes \mathrm{Id})(\pi(m)) = (\mathrm{Id} \otimes \Delta)(\pi(m)),\ (\mathrm{Id} \otimes \varepsilon)(\pi(m)) = m.$$

For example, C is a left and right comodule with $\pi = \Delta$, and so is k, with $\pi = \varepsilon$.

Exercise 1.20.3. (i) Show that if C is a coalgebra then C^* is an algebra, and if A is a finite dimensional algebra then A^* is a coalgebra.

(ii) Show that for any coalgebra C, any (left or right) C-comodule M is a (respectively, right or left) C^*-module, and the converse is true if C is finite dimensional.

Exercise 1.20.4. (i) Show that any coalgebra C is a sum of finite dimensional subcoalgebras.

Hint. Let $c \in C$, and let

$$(\Delta \otimes \mathrm{Id}) \circ \Delta(c) = (\mathrm{Id} \otimes \Delta) \circ \Delta(c) = \sum_i c_i^1 \otimes c_i^2 \otimes c_i^3.$$

Show that $\mathrm{span}(c_i^2)$ is a subcoalgebra of C containing c.

(ii) Show that any C-comodule is a sum of finite dimensional subcomodules.

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