1.25. **The Quantum Group $U_{\mathsf{q}}(\mathfrak{sl}_2)$.** Let us consider the Lie algebra $\mathfrak{sl}_2$. Recall that there is a basis $\mathsf{h}, \mathsf{e}, \mathsf{f} \in \mathfrak{sl}_2$ such that $[\mathsf{h}, \mathsf{e}] = 2\mathsf{e}, [\mathsf{h}, \mathsf{f}] = -2\mathsf{f}, [\mathsf{e}, \mathsf{f}] = \mathsf{h}$. This motivates the following definition.

Definition 1.25.1. Let $\mathsf{q} \in k, \mathsf{q} \neq \pm 1$. The *quantum group* $U_{\mathsf{q}}(\mathfrak{sl}_2)$ is generated by elements E, F and an invertible element K with defining relations

$$K\mathsf{E}K^{-1} = \mathsf{q}^2\mathsf{E}, K\mathsf{F}K^{-1} = \mathsf{q}^{-2}\mathsf{F}, [\mathsf{E}, \mathsf{F}] = \frac{K - K^{-1}}{\mathsf{q} - \mathsf{q}^{-1}}.$$

Theorem 1.25.2. *There exists a unique Hopf algebra structure on $U_{\mathsf{q}}(\mathfrak{sl}_2)$, given by*
- $\Delta(K) = K \otimes K$ *(thus K is a grouplike element);*
- $\Delta(\mathsf{E}) = \mathsf{E} \otimes K + 1 \otimes \mathsf{E}$*;*
- $\Delta(\mathsf{F}) = \mathsf{F} \otimes 1 + K^{-1} \otimes \mathsf{F}$ *(thus E, F are skew-primitive elements).*

Exercise 1.25.3. Prove Theorem 1.25.2.

Remark 1.25.4. Heuristically, $K = \mathsf{q}^{\mathsf{h}}$, and thus

$$\lim_{\mathsf{q} \to 1} \frac{K - K^{-1}}{\mathsf{q} - \mathsf{q}^{-1}} = \mathsf{h}.$$

So in the limit $\mathsf{q} \to 1$, the relations of $U_{\mathsf{q}}(\mathfrak{sl}_2)$ degenerate into the relations of $U(\mathfrak{sl}_2)$, and thus $U_{\mathsf{q}}(\mathfrak{sl}_2)$ should be viewed as a Hopf algebra deformation of the enveloping algebra $U(\mathfrak{sl}_2)$. In fact, one can make this heuristic idea into a precise statement, see e.g. [K].

If q is a root of unity, one can also define a finite dimensional version of $U_{\mathsf{q}}(\mathfrak{sl}_2)$. Namely, assume that the order of q is an odd number ℓ. Let $u_{\mathsf{q}}(\mathfrak{sl}_2)$ be the quotient of $U_{\mathsf{q}}(\mathfrak{sl}_2)$ by the additional relations

$$\mathsf{E}^\ell = \mathsf{F}^\ell = K^\ell - 1 = 0.$$

Then it is easy to show that $u_{\mathsf{q}}(\mathfrak{sl}_2)$ is a Hopf algebra (with the coproduct inherited from $U_{\mathsf{q}}(\mathfrak{sl}_2)$). This Hopf algebra is called *the small quantum group* attached to $\mathfrak{sl}_2$.

1.26. **The quantum group $U_{\mathsf{q}}(\mathfrak{g})$.** The example of the previous subsection can be generalized to the case of any simple Lie algebra. Namely, let $\mathfrak{g}$ be a simple Lie algebra of rank r, and let $A = (a_{ij})$ be its Cartan matrix. Recall that there exist unique relatively prime positive integers $d_i, i = 1, \dots r$ such that $d_i a_{ij} = d_j a_{ji}$. Let $\mathsf{q} \in k, \mathsf{q} \neq \pm 1$.

Definition 1.26.1.
- The q-*analog* of n is

$$[n]_{\mathsf{q}} = \frac{\mathsf{q}^n - \mathsf{q}^{-n}}{\mathsf{q} - \mathsf{q}^{-1}}.$$

- The q-*analog of the factorial* is

$$[n]_{\mathsf{q}}! = \prod_{l=1}^{n}[l]_{\mathsf{q}} = \frac{(\mathsf{q}-\mathsf{q}^{-1})\cdots(\mathsf{q}^n-\mathsf{q}^{-n})}{(\mathsf{q}-\mathsf{q}^{-1})^n}.$$

Definition 1.26.2. The *quantum group* $U_{\mathsf{q}}(\mathfrak{g})$ is generated by elements $\mathsf{E}_i, \mathsf{F}_i$ and invertible elements K_i, with defining relations

$$K_iK_j = K_jK_i,\ K_i\mathsf{E}_jK_i^{-1} = \mathsf{q}^{a_{ij}}\mathsf{E}_j, K_i\mathsf{F}_jK_i^{-1} = \mathsf{q}^{-a_{ij}}\mathsf{F}_j,$$

$$[\mathsf{E}_i, \mathsf{F}_j] = \delta_{ij}\frac{K_i^{d_i}-K_i^{-d_i}}{\mathsf{q}^{d_i}-\mathsf{q}^{-d_i}}, \quad \text{and the } \mathsf{q}\text{-Serre relations:}$$

$$\sum_{l=0}^{1-a_{ij}} \frac{(-1)^l}{[l]_{\mathsf{q}_i}![1-a_{ij}-l]_{\mathsf{q}_i}!}\mathsf{E}_i^{1-a_{ij}-l}\mathsf{E}_j\mathsf{E}_i^l = 0, i \neq j \tag{1.26.1}$$

and

$$\sum_{l=0}^{1-a_{ij}} \frac{(-1)^l}{[l]_{\mathsf{q}_i}![1-a_{ij}-l]_{\mathsf{q}_i}!}\mathsf{F}_i^{1-a_{ij}-l}\mathsf{F}_j\mathsf{F}_i^l = 0, i \neq j. \tag{1.26.2}$$

More generally, the same definition can be made for any symmetrizable Kac-Moody algebra $\mathfrak{g}$.

Theorem 1.26.3. *(see e.g.* [CP]*) There exists a unique Hopf algebra structure on $U_{\mathsf{q}}(\mathfrak{g})$, given by*

- $\Delta(K_i) = K_i \otimes K_i$;
- $\Delta(\mathsf{E}_i) = \mathsf{E}_i \otimes K_i + 1 \otimes \mathsf{E}_i$;
- $\Delta(\mathsf{F}_i) = \mathsf{F}_i \otimes 1 + K_i^{-1} \otimes \mathsf{F}_i$.

Remark 1.26.4. Similarly to the case of $\mathfrak{sl}_2$, in the limit $\mathsf{q} \to 1$, these relations degenerate into the relations for $U(\mathfrak{g})$, so $U_{\mathsf{q}}(\mathfrak{g})$ should be viewed as a Hopf algebra deformation of the enveloping algebra $U(\mathfrak{g})$.

1.27. **Categorical meaning of skew-primitive elements.** We have seen that many interesting Hopf algebras contain nontrivial skew-primitive elements. In fact, the notion of a skew-primitive element has a categorical meaning. Namely, we have the following proposition.

Proposition 1.27.1. *Let g, h be grouplike elements of a coalgebra C, and $\mathrm{Prim}_{h,g}(C)$ be the space of skew-primitive elements of type h, g. Then the space $\mathrm{Prim}_{h,g}(H)/k(h-g)$ is naturally isomorphic to $\mathrm{Ext}^1(g, h)$, where g, h are regarded as 1-dimensional right C-comodules.*

Proof. Let V be a 2-dimensional H-comodule, such that we have an exact sequence

$$0 \to h \to V \to g \to 0.$$

Then V has a basis v_0, v_1 such that

$$\pi(v_0) = v_0 \otimes h,\ \pi(v_1) = v_1 \otimes x + v_0 \otimes g.$$

The condition that this is a comodule yields that x is a skew-primitive element of type (h, g). So any extension defines a skew-primitive element and vice versa. Also, we can change the basis by $v_0 \to v_0$, $v_1 \to v_1 + \lambda v_0$, which modifies x by adding a trivial skew-primitive element. This implies the result. □

Example 1.27.2. The category $\mathcal{C}$ of finite dimensional comodules over $u_{\mathfrak{q}}(\mathfrak{sl}_2)$ is an example of a finite tensor category in which there are objects V such that V^{**} is not isomorphic to V. Namely, in this category, the functor $V \mapsto V^{**}$ is defined by the squared antipode S^2, which is conjugation by K: $S^2(x) = KxK^{-1}$. Now, we have $\mathrm{Ext}^1(K, 1) = Y = \langle E, FK \rangle$, a 2-dimensional space. The set of isomorphism classes of nontrivial extensions of K by 1 is therefore the projective line $\mathbb{P}Y$. The operator of conjugation by K acts on Y with eigenvalues q^2, q^{-2}, hence nontrivially on $\mathbb{P}Y$. Thus for a generic extension V, the object V^{**} is not isomorphic to V.

However, note that *some power* of the functor $**$ on $\mathcal{C}$ is isomorphic (in fact, monoidally) to the identity functor (namely, this power is the order of q). We will later show that this property holds in any finite tensor category.

Note also that in the category $\mathcal{C}$, $V^{**} \cong V$ if V is simple. This clearly has to be the case in any tensor category where all simple objects are invertible. We will also show (see Proposition 1.41.1 below) that this is the case in any semisimple tensor category. An example of a tensor category in which V^{**} is not always isomorphic to V even for simple V is the category of finite dimensional representations of the the Yangian $H = Y(\mathfrak{g})$ of a simple complex Lie algebra $\mathfrak{g}$, see [CP, 12.1]. Namely, for any finite dimensional representation V of H and any complex number z one can define the shifted representation $V(z)$ (such that $V(0) = V$). Then $V^{*}{*} \cong V(2h^\vee)$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, see [CP, p.384]. If V is a non-trivial irreducible finite dimensional representation then $V(z) \ncong V$ for $z \neq 0$. Thus, $V^{**} \ncong V$. Moreover, we see that the functor $**$ has infinite order even when restricted to simple objects of $\mathcal{C}$.

However, the representation category of the Yangian is infinite, and the answer to the following question is unknown to us.

Question 1.27.3. Does there exist a *finite* tensor category, in which there is a simple object V such that V^{**} is not isomorphic to V? (The answer is unknown to the authors).

Theorem 1.27.4. *Assume that k has characteristic 0. Let $\mathcal{C}$ be a finite ring category over k with simple object $\mathbf{1}$. Then* $\mathrm{Ext}^1(\mathbf{1},\mathbf{1}) = 0$.

Proof. Assume the contrary, and suppose that V is a nontrivial extension of $\mathbf{1}$ by itself. Let P be the projective cover of $\mathbf{1}$. Then $\mathsf{Hom}(P, V)$ is a 2-dimensional space, with a filtration induced by the filtration on V, and both quotients naturally isomorphic to $E := \mathsf{Hom}(P, \mathbf{1})$. Let v_0, v_1 be a basis of $\mathsf{Hom}(P, V)$ compatible to the filtration, i.e. v_0 spans the 1-dimensional subspace defined by the filtration. Let $A = \mathsf{End}(P)$ (this is a finite dimensional algebra). Let $\varepsilon : A \to k$ be the character defined by the (right) action of A on E. Then the matrix of $a \in A$ in the basis v_0, v_1 has the form

$$[a]_1 = \begin{pmatrix} \varepsilon(a) & \chi_1(a) \\ 0 & \varepsilon(a) \end{pmatrix} \tag{1.27.1}$$

where $\chi_1 \in A^*$ is nonzero. Since $a \to [a]_1$ is a homomorphism, χ_1 is a derivation: $\chi_1(xy) = \chi_1(x)\varepsilon(y) + \varepsilon(x)\chi_1(y)$.

Now consider the representation $V \otimes V$. Using the exactness of tensor products, we see that the space $\mathsf{Hom}(P, V \otimes V)$ is 4-dimensional, and has a 3-step filtration, with successive quotients $E, E \oplus E, E$, and basis $v_{00}; v_{01}, v_{10}; v_{11}$, consistent with this filtration. The matrix of $a \in \mathsf{End}(P)$ in this basis is

$$[a]_2 = \begin{pmatrix} \varepsilon(a) & \chi_1(a) & \chi_1(a) & \chi_2(a) \\ 0 & \varepsilon(a) & 0 & \chi_1(a) \\ 0 & 0 & \varepsilon(a) & \chi_1(a) \\ 0 & 0 & 0 & \varepsilon(a) \end{pmatrix} \tag{1.27.2}$$

Since $a \to [a]_2$ is a homomorphism, we find

$$\chi_2(ab) = \varepsilon(a)\chi_2(b) + \chi_2(a)\varepsilon(b) + 2\chi_1(a)\chi_1(b).$$

We can now proceed further (i.e. consider $V \otimes V \otimes V$ etc.) and define for every positive n, a linear function $\chi_n \in A^*$ which satisfies the equation

$$\chi_n(ab) = \sum_{j=0}^{n} \binom{n}{j} \chi_j(a)\chi_{n-j}(b),$$

where $\chi_0 = \varepsilon$. Thus for any $s \in k$, we can define $\phi_s : A \to k((t))$ by $\phi_s(a) = \sum_{m \geq 0} \chi_m(a) s^m t^m / m!$, and we find that ϕ_s is a family of pairwise distinct homomorphisms. This is a contradiction, as A is a finite dimensional algebra. We are done. □

Corollary 1.27.5. *If a finite ring category $\mathcal{C}$ over a field of characteristic zero has a unique simple object $\mathbf{1}$, then $\mathcal{C}$ is equivalent to the category Vec.*

Corollary 1.27.6. *A finite dimensional bialgebra H over a field of characteristic zero cannot contain nonzero primitive elements.*

Proof. Apply Theorem 1.27.4 to the category $H-$ comod and use Proposition 1.27.1. □

Remark 1.27.7. Here is a "linear algebra" proof of this corollary. Let x be a nonzero primitive element of H. Then we have a family of grouplike elements $e^{stx} \in H((t))$, $s \in k$, i.e., an infinite collection of characters of $H^*((t))$, which is impossible, as H is finite dimensional.

Corollary 1.27.8. *If H is a finite dimensional* commutative *Hopf algebra over an algebraically closed field k of characteristic 0, then $H = \mathrm{Fun}(G, k)$ for a unique finite group G.*

Proof. Let $G = \mathrm{Spec}(H)$ (a finite group scheme), and $x \in T_1G = (\mathfrak{m}/\mathfrak{m}^2)^*$ where $\mathfrak{m}$ is the kernel of the counit. Then x is a linear function on $\mathfrak{m}$. Extend it to H by setting $x(1) = 0$. Then x s a derivation:

$$x(fg) = x(f)g(1) + f(1)x(g).$$

This implies that x is a primitive element in H^*. So by Corollary 1.27.6, $x = 0$. this implies that G is reduced at the point 1. By using translations, we see that G is reduced at all other points. So G is a finite group, and we are done. □

Remark 1.27.9. Theorem 1.27.4 and all its corollaries fail in characteristic $p > 0$. A counterexample is provided by the Hopf algebra $k[x]/(x^p)$, where x is a primitive element.

1.28. **Pointed tensor categories and pointed Hopf algebras.**

Definition 1.28.1. A coalgebra C is *pointed* if its category of right comodules is pointed, i.e., if any simple right C-comodule is 1-dimensional.

Remark 1.28.2. A finite dimensional coalgebra C is pointed if and only if the algebra C^* is basic, i.e., the quotient $C^*/\mathrm{Rad}(C^*)$ of C^* by its radical is commutative. In this case, simple C-comodules are points of $\mathrm{Specm}(H^*/\mathrm{Rad}(H^*))$, which justifies the term "pointed".

Definition 1.28.3. A tensor category $\mathcal{C}$ is pointed if every simple object of $\mathcal{C}$ is invertible.

Thus, the category of right comodules over a Hopf algebra H is pointed if and only if H is pointed.

Example 1.28.4. The category Vec_G^ω is a pointed category. If G is a p-group and k has characteristic p, then $\mathrm{Rep}_k(G)$ is pointed. Any cocommutative Hopf algebra, the Taft and Nichols Hopf algebras, as well as the quantum groups $U_\mathfrak{q}(\mathfrak{g})$ are pointed Hopf algebras.

1.29. **The coradical filtration.** Let $\mathcal{C}$ be a locally finite abelian category.

Any object $X \in \mathcal{C}$ has a canonical filtration

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X \tag{1.29.1}$$

such that X_{i+1}/X_i is the socle (i.e., the maximal semisimple subobject) of X/X_i (in other words, X_{i+1}/X_i is the sum of all simple subobjects of X/X_i). This filtration is called the *socle filtration*, or the *coradical filtration* of X. It is easy to show by induction that the coradical filtration is a filtration of X of the smallest possible length, such that the successive quotients are semisimple. The length of the coradical filtration of X is called the *Loewy length* of X, and denoted $\mathrm{Lw}(X)$. Then we have a filtration of the category $\mathcal{C}$ by Loewy length of objects: $\mathcal{C}_0 \subset \mathcal{C}_1 \subset ...$, where $\mathcal{C}_i$ denotes the full subcategory of objects of $\mathcal{C}$ of Loewy length $\le i+1$. Clearly, the Loewy length of any subquotient of an object X does not exceed the Loewy length of X, so the categories $\mathcal{C}_i$ are closed under taking subquotients.

Definition 1.29.1. The filtration of $\mathcal{C}$ by $\mathcal{C}_i$ is called the *coradical filtration* of $\mathcal{C}$.

If $\mathcal{C}$ is endowed with an exact faithful functor $F : \mathcal{C} \to \mathrm{Vec}$ then we can define the coalgebra $C = \mathrm{Coend}(F)$ and its subcoalgebras $C_i = \mathrm{Coend}(F|_{\mathcal{C}_i})$, and we have $C_i \subset C_{i+1}$ and $C = \cup_i C_i$ (alternatively, we can say that C_i is spanned by matrix elements of C-comodules $F(X)$, $X \in \mathcal{C}_i$). Thus we have defined an increasing filtration by subcoalgebras of any coalgebra C. This filtration is called the *coradical filtration*, and the term C_0 is called the *coradical* of C.

The "linear algebra" definition of the coradical filtration is as follows. One says that a coalgebra is *simple* if it does not have nontrivial subcoalgebras, i.e. if it is finite dimensional, and its dual is a simple (i.e., matrix) algebra. Then C_0 is the sum of all simple subcoalgebras of C. The coalgebras C_{n+1} for $n \ge 1$ are then defined inductively to be the spaces of those $x \in C$ for which

$$\Delta(x) \in C_n \otimes C + C \otimes C_0.$$

Exercise 1.29.2. (i) Suppose that C is a finite dimensional coalgebra, and I is the Jacobson radical of C^*. Show that $C_n^\perp = I^{n+1}$. This justifies the term "coradical filtration".

(ii) Show that the coproduct respects the coradical filtration, i.e. $\Delta(C_n) \subset \sum_{i=0}^n C_i \otimes C_{n-i}$.

(iii) Show that C_0 is the **direct** sum of simple subcoalgebras of C. In particular, grouplike elements of any coalgebra C are linearly independent.

Hint. Simple subcoalgebras of C correspond to finite dimensional irrreducible representations of C^*.

Denote by $\mathrm{gr}(C)$ the associated graded coalgebra of a coalgebra C with respect to the coradical filtration. Then $\mathrm{gr}(C)$ is a $\mathbb{Z}_+$-graded coalgebra. It is easy to see from Exercise 1.29.2(i) that the coradical filtration of $\mathrm{gr}(C)$ is induced by its grading. A graded coalgebra $\bar{C}$ with this property is said to be *coradically graded*, and a coalgebra C such that $\mathrm{gr}(C) = \bar{C}$ is called a *lifting* of C.

Definition 1.29.3. A coalgebra C is said to be *cosemisimple* if C is a direct sum of simple subcoalgebras.

Clearly, C is cosemisimple if and only if $C -$ comod is a semisimple category.

Proposition 1.29.4. *(i) A category $\mathcal{C}$ is semisimple if and only if $\mathcal{C}_0 = \mathcal{C}_1$.*

(ii) A coalgebra C is cosemisimple if and only if $C_0 = C_1$.

Proof. (ii) is a special case of (i), and (i) is clear, since the condition means that $\mathrm{Ext}^1(X, Y) = 0$ for any simple X, Y, which implies (by the long exact sequence of cohomology) that $\mathrm{Ext}^1(X, Y) = 0$ for all $X, Y \in \mathcal{C}$. □

Corollary 1.29.5. *(The Taft-Wilson theorem) If C is a pointed coalgebra, then C_0 is spanned by (linearly independent) grouplike elements g, and $C_1/C_0 = \oplus_{h,g}\mathrm{Prim}_{h,g}(C)/k(h - g)$. In particular, any non-cosemisimple pointed coalgebra contains nontrivial skew-primitive elements.*

Proof. The first statement is clear (the linear independence follows from Exercise 1.29.2(iii). Also, it is clear that any skew-primitive element is contained in C_1. Now, if $x \in C_1$, then x is a matrix element of a C-comodule of Loewy length ≤ 2, so it is a sum of matrix elements 2-dimensional comodules, i.e. of grouplike and skew-primitive elements.

It remains to show that the sum $\sum_{h,g} \mathrm{Prim}_{h,g}(C)/k(h - g) \subset C/C_0$ is direct. For this, it suffices to consider the case when C is finite dimensional. Passing to the dual algebra $A = C^*$, we see that the statement is equivalent to the claim that I/I^2 (where I is the radical of A) is isomorphic (in a natural way) to $\oplus_{g,h}\mathrm{Ext}^1(g, h)^*$.

Let p_g be a complete system of orthogonal idempotents in A/I^2, such that $h(p_g) = \delta_{hg}$. Define a pairing $I/I^2 \times \mathrm{Ext}^1(g, h) \to k$ which sends $a \otimes \alpha$ to the upper right entry of the 2-by-2 matrix by which a acts in the extension of g by h defined by α. It is easy to see that this pairing descends to a pairing $B : p_h(I/I^2)p_g \times \mathrm{Ext}^1(g,h) \to k$. If the extension α is nontrivial, the upper right entry cannot be zero, so B is right-nondegenerate. Similarly, if a belongs to the left kernel of B, then a acts by zero in any A-module of Loewy length 2, so $a = 0$. Thus, B is left-nondegenerate. This implies the required isomorphism. □

Proposition 1.29.6. *If C, D are coalgebras, and $f : C \to D$ is a coalgebra homomorphism such that $f|_{C_1}$ is injective, then f is injective.*

Proof. One may assume that C and D are finite dimensional. Then the statement can be translated into the following statement about finite dimensional algebras: if A, B are finite dimensional algebras and $f : A \to B$ is an algebra homomorphism which descends to a surjective homomorphism $A \to B/\mathrm{Rad}(B)^2$, then f is surjective.

To prove this statement, let $b \in B$. Let $I = \mathrm{Rad}(B)$. We prove by induction in n that there exists $a \in A$ such that $b - f(a) \in I^n$. The base of induction is clear, so we only need to do the step of induction. So assume $b \in I^n$. We may assume that $b = b_1...b_n$, $b_i \in I$, and let $a_i \in A$ be such that $f(a_i) = b_i$ modulo I^2. Let $a = a_1...a_n$. Then $b - f(a) \in I^{n+1}$, as desired. □

Corollary 1.29.7. *If H is a Hopf algebra over a field of characteristic zero, then the natural map $\xi : U(\mathrm{Prim}(H)) \to H$ is injective.*

Proof. By Proposition 1.29.6, it suffices to check the injectivity of ξ in degree 1 of the coradical filtration. Thus, it is enough to check that ξ is injective on primitive elements of $U(\mathrm{Prim}(H))$. But by Exercise 1.24.4, all of them lie in $\mathrm{Prim}(H)$, as desired. □

1.30. **Chevalley's theorem.**

Theorem 1.30.1. *(Chevalley) Let k be a field of characteristic zero. Then the tensor product of two simple finite dimensional representations of any group or Lie algebra over k is semisimple.*

Proof. Let V be a finite dimensional vector space over a field k (of any characteristic), and $G \subset GL(V)$ be a Zariski closed subgroup.

Lemma 1.30.2. *If V is a completely reducible representation of G, then G is reductive.*

Proof. Let V be a nonzero rational representation of an affine algebraic group G. Let U be the unipotent radical of G. Let $V^U \subset V$ be the space of invariants. Since U is unipotent, $V^U \neq 0$. So if V is irreducible, then $V^U = V$, i.e., U acts trivially. Thus, U acts trivially on any completely reducible representation of G. So if V is completely reducible and $G \subset GL(V)$, then G is reductive. □

Now let G be any group, and V, W be two finite dimensional irreducible representations of G. Let G_V, G_W be the Zariski closures of the images of G in $GL(V)$ and $GL(W)$, respectively. Then by Lemma 1.30.2 G_V, G_W are reductive. Let G_{VW} be the Zariski closure of the image of G in $G_V \times G_W$. Let U be the unipotent radical of G_{VW}. Let $p_V : G_{VW} \to G_V$, $p_W : G_{VW} \to G_W$ be the projections. Since p_V is surjective, $p_V(U)$ is a normal unipotent subgroup of G_V, so $p_V(U) = 1$. Similarly, $p_W(U) = 1$. So $U = 1$, and G_{VW} is reductive.

Let G'_{VW} be the closure of the image of G in $GL(V \otimes W)$. Then G'_{VW} is a quotient of G_{VW}, so it is also reductive. Since $\mathrm{char} k = 0$, this implies that the representation $V \otimes W$ is completely reducible as a representation of G'_{VW}, hence of G.

This proves Chevalley's theorem for groups. The proof for Lie algebras is similar. □

1.31. **Chevalley property.**

Definition 1.31.1. A tensor category $\mathcal{C}$ is said to have *Chevalley property* if the category $\mathcal{C}_0$ of semisimple objects of $\mathcal{C}$ is a tensor subcategory.

Thus, Chevalley theorem says that the category of finite dimensional representations of any group or Lie algebra over a field of characteristic zero has Chevalley property.

Proposition 1.31.2. *A pointed tensor category has Chevalley property.*

Proof. Obvious. □

Proposition 1.31.3. *In a tensor category with Chevalley property,*

$$\mathrm{Lw}(X \otimes Y) \le \mathrm{Lw}(X) + \mathrm{Lw}(Y) - 1. \tag{1.31.1}$$

Thus $\mathcal{C}_i \otimes \mathcal{C}_j \subset \mathcal{C}_{i+j}$.

Proof. Let $X(i)$, $0 \le i \le m$, $Y(j)$, $0 \le j \le n$, be the successive quotients of the coradical filtrations of X, Y. Then $Z := X \otimes Y$ has a filtration with successive quotients $Z(r) = \oplus_{i+j=r} X(i) \otimes Y(j)$, $0 \le r \le m+n$. Because of the Chevalley property, these quotients are semisimple. This implies the statement. □

Remark 1.31.4. It is clear that the converse to Proposition 1.31.3 holds as well: equation (1.31.3) (for simple X and Y) implies the Chevalley property.

Corollary 1.31.5. *In a pointed Hopf algebra H, the coradical filtration is a Hopf algebra filtration, i.e.* $H_i H_j \subset H_{i+j}$ *and* $S(H_i) = H_i$, *so* $\mathrm{gr}(H)$ *is a Hopf algebra.*

In this situation, the Hopf algebra H is said to be a *lifting* of the coradically graded Hopf algebra $\mathrm{gr}(H)$.

Example 1.31.6. The Taft algebra and the Nichols algebras are coradically graded. The associated graded Hopf algebra of $U_{\mathsf{q}}(\mathfrak{g})$ is the Hopf algebra defined by the same relations as $U_{\mathsf{q}}(\mathfrak{g})$, except that the commutation relation between E_i and F_j is replaced with the condition that E_i and F_j commute (for all i, j). The same applies to the small quantum group $u_{\mathsf{q}}(\mathfrak{sl}_2)$.

Exercise 1.31.7. Let k be a field of characteristic p, and G a finite group. Show that the category $\mathrm{Rep}_k(G)$ has Chevalley property if and only if G has a normal p-Sylow subgroup.