1.45. **Tensor categories with finitely many simple objects. Frobenius-Perron dimensions.** Let A be a $\mathbb{Z}_+$-ring with $\mathbb{Z}_+$-basis I.

Definition 1.45.1. We will say that A is *transitive* if for any $X, Z \in I$ there exist $Y_1, Y_2 \in I$ such that XY_1 and Y_2X involve Z with a nonzero coefficient.

Proposition 1.45.2. *If $\mathcal{C}$ is a ring category with right duals then $\mathsf{Gr}(\mathcal{C})$ is a transitive unital $\mathbb{Z}_+$-ring.*

Proof. Recall from Theorem 1.15.8 that the unit object $\mathbf{1}$ in $\mathcal{C}$ is simple. So $\mathsf{Gr}(\mathcal{C})$ is unital. This implies that for any simple objects X, Z of $\mathcal{C}$, the object $X \otimes X^* \otimes Z$ contains Z as a composition factor (as $X \otimes X^*$ contains $\mathbf{1}$ as a composition factor), so one can find a simple object Y_1 occurring in $X^* \otimes Z$ such that Z occurs in $X \otimes Y_1$. Similarly, the object $Z \otimes X^* \otimes X$ contains Z as a composition factor, so one can find a simple object Y_2 occurring in $Z \otimes X^*$ such that Z occurs in $Y_2 \otimes X$. Thus $\mathsf{Gr}(\mathcal{C})$ is transitive. ☐

Let A be a transitive unital $\mathbb{Z}_+$-ring of finite rank. Define the group homomorphism $\mathsf{FPdim} : A \to \mathbb{C}$ as follows. For $X \in I$, let $\mathsf{FPdim}(X)$ be the maximal nonnegative eigenvalue of the matrix of left multiplication by X. It exists by the Frobenius-Perron theorem, since this matrix has nonnegative entries. Let us extend FPdim from the basis I to A by additivity.

Definition 1.45.3. The function FPdim is called the *Frobenius-Perron dimension.*

In particular, if $\mathcal{C}$ is a ring category with right duals and finitely many simple objects, then we can talk about Frobenius-Perron dimensions of objects of $\mathcal{C}$.

Proposition 1.45.4. *Let $X \in I$.*

(1) *The number $\alpha = \mathsf{FPdim}(X)$ is an algebraic integer, and for any algebraic conjugate α' of α we have $\alpha \geq |\alpha'|$.*
(2) $\mathsf{FPdim}(X) \geq 1$.

Proof. (1) Note that α is an eigenvalue of the integer matrix N_X of left multiplication by X, hence α is an algebraic integer. The number α' is a root of the characteristic polynomial of N_X, so it is also an eigenvalue of N_X. Thus by the Frobenius-Perron theorem $\alpha \geq |\alpha'|$.

(2) Let r be the number of algebraic conjugates of α. Then $\alpha^r \geq N(\alpha)$ where $N(\alpha)$ is the norm of α. This implies the statement since $N(\alpha) \geq 1$. ☐

Proposition 1.45.5. (1) *The function* $\mathsf{FPdim} : A \to \mathbb{C}$ *is a ring homomorphism.*

(2) *There exists a unique, up to scaling, element* $R \in A_{\mathbb{C}} := A \otimes_{\mathbb{Z}} \mathbb{C}$ *such that* $XR = \mathsf{FPdim}(X)R$, *for all* $X \in A$. *After an appropriate normalization this element has positive coefficients, and satisfies* $\mathsf{FPdim}(R) > 0$ *and* $RY = \mathsf{FPdim}(Y)R$, $Y \in A$.

(3) FPdim *is a unique nonzero character of* A *which takes nonnegative values on* I.

(4) *If* $X \in A$ *has nonnegative coefficients with respect to the basis of* A, *then* $\mathsf{FPdim}(X)$ *is the largest nonnegative eigenvalue* $\lambda(N_X)$ *of the matrix* N_X *of multiplication by* X.

Proof. Consider the matrix M of right multiplication by $\sum_{X \in I} X$ in A in the basis I. By transitivity, this matrix has strictly positive entries, so by the Frobenius-Perron theorem, part (2), it has a unique, up to scaling, eigenvector $R \in A_{\mathbb{C}}$ with eigenvalue $\lambda(M)$ (the maximal positive eigenvalue of M). Furthermore, this eigenvector can be normalized to have strictly positive entries.

Since R is unique, it satisfies the equation $XR = d(X)R$ for some function $d : A \to \mathbb{C}$. Indeed, XR is also an eigenvector of M with eigenvalue $\lambda(M)$, so it must be proportional to R. Furthermore, it is clear that d is a character of A. Since R has positive entries, $d(X) = \mathsf{FPdim}(X)$ for $X \in I$. This implies (1). We also see that $\mathsf{FPdim}(X) > 0$ for $X \in I$ (as R has strictly positive coefficients), and hence $\mathsf{FPdim}(R) > 0$.

Now, by transitivity, R is the unique, up to scaling, solution of the system of linear equations $XR = \mathsf{FPdim}(X)R$ (as the matrix N of left multiplication by $\sum_{X \in I} X$ also has positive entries). Hence, $RY = d'(Y)R$ for some character d'. Applying FPdim to both sides and using that $\mathsf{FPdim}(R) > 0$, we find $d' = \mathsf{FPdim}$, proving (2).

If χ is another character of A taking positive values on I, then the vector with entries $\chi(Y)$, $Y \in I$ is an eigenvector of the matrix N of the left multiplication by the element $\sum_{X \in I} X$. Because of transitivity of A the matrix N has positive entries. By the Frobenius-Perron theorem there exists a positive number λ such that $\chi(Y) = \lambda \, \mathsf{FPdim}(Y)$. Since χ is a character, $\lambda = 1$, which completes the proof.

Finally, part (4) follows from part (2) and the Frobenius-Perron theorem (part (3)). □

Example 1.45.6. Let $\mathcal{C}$ be the category of finite dimensional representations of a quasi-Hopf algebra H, and A be its Grothendieck ring. Then by Proposition 1.10.9, for any $X, Y \in \mathcal{C}$

$$\dim \mathsf{Hom}(X \otimes H, Y) = \dim \mathsf{Hom}(H, {}^*X \otimes Y) = \dim(X)\dim(Y),$$

where H is the regular representation of H. Thus $X \otimes H = \dim(X)H$, so $\mathsf{FPdim}(X) = \dim(X)$ for all X, and $R = H$ up to scaling.

This example motivates the following definition.

Definition 1.45.7. The element R will be called a *regular element* of A.

Proposition 1.45.8. *Let A be as above and $* : I \to I$ be a bijection which extends to an anti-automorphism of A. Then FPdim is invariant under $*$.*

Proof. Let $X \in I$. Then the matrix of right multiplication by X^* is the transpose of the matrix of left multiplication by X modified by the permutation $*$. Thus the required statement follows from Proposition 1.45.5(2). □

Corollary 1.45.9. *Let $\mathcal{C}$ be a ring category with right duals and finitely many simple objects, and let X be an object in $\mathcal{C}$. If $\mathsf{FPdim}(X) = 1$ then X is invertible.*

Proof. By Exercise 1.15.10(d) it is sufficient to show that $X \otimes X^* = \mathbf{1}$. This follows from the facts that $\mathbf{1}$ is contained in $X \otimes X^*$ and $\mathsf{FPdim}(X \otimes X^*) = \mathsf{FPdim}(X)\,\mathsf{FPdim}(X^*) = 1$. □

Proposition 1.45.10. *Let $f : A_1 \to A_2$ be a unital homomorphism of transitive unital $\mathbb{Z}_+$-rings of finite rank, whose matrix in their $\mathbb{Z}_+$-bases has non-negative entries. Then*

1. *f preserves Frobenius-Perron dimensions.*
2. *Let I_1, I_2 be the $\mathbb{Z}_+$-bases of A_1, A_2, and suppose that for any Y in I_2 there exists $X \in I_1$ such that the coefficient of Y in $f(X)$ is non-zero. If R is a regular element of A_1 then $f(R)$ is a regular element of A_2.*

Proof. (1) The function $X \mapsto \mathsf{FPdim}(f(X))$ is a nonzero character of A_1 with nonnegative values on the basis. By Proposition 1.45.5(3), $\mathsf{FPdim}(f(X)) = \mathsf{FPdim}(X)$ for all X in I. (2) By part (1) we have

$$f(\sum_{X \in I_1} X) f(R_1) = \mathsf{FPdim}(f(\sum_{X \in I_1} X)) f(R_1). \tag{1.45.1}$$

But $f(\sum_{X \in I_1} X)$ has strictly positive coefficients in I_2, hence $f(R_1) = \beta R_2$ for some $\beta > 0$. Applying FPdim to both sides, we get the result. □

Corollary 1.45.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be tensor categories with finitely many classes of simple objects. If $F : \mathcal{C} \to \mathcal{D}$ be a quasi-tensor functor, then $\mathsf{FPdim}_{\mathcal{D}}(F(X)) = \mathsf{FPdim}_{\mathcal{C}}(X)$ for any X in $\mathcal{C}$.*

Example 1.45.12. (Tambara-Yamagami fusion rings) Let G be a finite group, and TY_G be an extension of the unital based ring $\mathbb{Z}[G]$:

$$TY_G := \mathbb{Z}[G] \oplus \mathbb{Z}X,$$

where X is a new basis vector with $gX = Xg = X$, $X^2 = \sum_{g\in G} g$. This is a fusion ring, with $X^* = X$. It is easy to see that $\mathsf{FPdim}(g) = 1$, $\mathsf{FPdim}(X) = |G|^{1/2}$. We will see later that these rings are categorifiable if and only if G is abelian.

Example 1.45.13. (Verlinde rings for $\mathfrak{sl}_2$). Let k be a nonnegative integer. Define a unital $\mathbb{Z}_+$-ring $\mathrm{Ver}_k = \mathrm{Ver}_k(\mathfrak{sl}_2)$ with basis V_i, $i = 0, ..., k$ ($V_0 = 1$), with duality given by $V_i^* = V_i$ and multiplication given by the truncated Clebsch-Gordan rule:

$$V_i \otimes V_j = \bigoplus_{l=|i-j|,\, i+j-l\in 2\mathbb{Z}}^{\min(i+j,2k-(i+j))} V_l. \tag{1.45.2}$$

It other words, one computes the product by the usual Clebsch-Gordan rule, and then deletes the terms that are not defined (V_i with $i > k$) and also their mirror images with respect to point $k+1$. We will show later that this ring admits categorifications coming from quantum groups at roots of unity.

Note that $\mathrm{Ver}_0 = \mathbb{Z}$, $\mathrm{Ver}_1 = \mathbb{Z}[\mathbb{Z}_2]$, $\mathrm{Ver}_2 = TY_{\mathbb{Z}_2}$. The latter is called the *Ising fusion ring*, as it arises in the Ising model of statistical mechanics.

Exercise 1.45.14. Show that $\mathsf{FPdim}(V_j) = [j+1]_q := \frac{q^{j+1}-q^{-j-1}}{q-q^{-1}}$, where $q = e^{\frac{\pi i}{k+2}}$.

Note that the Verlinde ring has a subring Ver_k^0 spanned by V_j with even j. If $k = 3$, this ring has basis $1, X = V_2$ with $X^2 = X + 1, X^* = X$. This ring is called the *Yang-Lee* fusion ring. In the Yang-Lee ring, $\mathsf{FPdim}(X)$ is the golden ratio $\frac{1+\sqrt{5}}{2}$.

Note that one can define the generalized Yang-Lee fusion rings YL_n $n \in \mathbb{Z}_+$, with basis $1, X$, multiplication $X^2 = 1+nX$ and duality $X^* = X$. It is, however, shown in [O2] that these rings are not categorifiable when $n > 1$.

Proposition 1.45.15. *(Kronecker) Let B be a matrix with nonnegative integer entries, such that $\lambda(BB^T) = \lambda(B)^2$. If $\lambda(B) < 2$ then $\lambda(B) = 2\cos(\pi/n)$ for some integer $n \geq 2$.*

Proof. Let $\lambda(B) = q + q^{-1}$. Then q is an algebraic integer, and $|q| = 1$. Moreover, all conjugates of $\lambda(B)^2$ are nonnegative (since they are eigenvalues of the matrix BB^T, which is symmetric and nonnegative definite), so all conjugates of $\lambda(B)$ are real. Thus, if q_* is a conjugate of q then $q_* + q_*^{-1}$ is real with absolute value < 2 (by the Frobenius-Perron theorem), so $|q_*| = 1$. By a well known result in elementary algebraic number theory, this implies that q is a root of unity: $q = e^{2\pi ik/m}$, where k and m are coprime. By the Frobenius-Perron theorem, so $k = \pm 1$, and m is even (indeed, if $m = 2p + 1$ is odd then $|q^p + q^{-p}| > |q + q^{-1}|$). So $q = e^{\pi i/n}$ for some integer $n \geq 2$, and we are done. $\square$

Corollary 1.45.16. *Let A be a fusion ring, and $X \in A$ a basis element. Then if $FPdim(X) < 2$ then $FPdim(X) = 2\cos(\pi/n)$, for some integer $n \geq 3$.*

Proof. This follows from Proposition 1.45.15, since $FPdim(XX^*) = FPdim(X)^2$. $\square$

1.46. **Deligne's tensor product of finite abelian categories.** Let $\mathcal{C}, \mathcal{D}$ be two finite abelian categories over a field k.

Definition 1.46.1. *Deligne's tensor product* $\mathcal{C} \boxtimes \mathcal{D}$ is an abelian category which is universal for the functor assigning to every k-linear abelian category $\mathcal{A}$ the category of right exact in both variables bilinear bifunctors $\mathcal{C} \times \mathcal{D} \to \mathcal{A}$. That is, there is a bifunctor $\boxtimes : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D} : (X, Y) \mapsto X \boxtimes Y$ which is right exact in both variables and is such that for any right exact in both variables bifunctor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$ there exists a unique right exact functor $\bar{F} : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{A}$ satisfying $\bar{F} \circ \boxtimes = F$.

Proposition 1.46.2. *(cf.* [D, Proposition 5.13]*) (i) The tensor product $\mathcal{C} \boxtimes \mathcal{D}$ exists and is a finite abelian category.*

(ii) It is unique up to a unique equivalence.

(iii) Let C, D be finite dimensional algebras and let $\mathcal{C} = C - \text{mod}$ and $\mathcal{D} = D - \text{mod}$. Then $\mathcal{C} \boxtimes \mathcal{D} = C \otimes D - \text{mod}$.

(iv) The bifunctor $\boxtimes$ is exact in both variables and satisfies

$$\mathsf{Hom}_{\mathcal{C}}(X_1, Y_1) \otimes \mathsf{Hom}_{\mathcal{D}}(X_2, Y_2) \cong \mathsf{Hom}_{\mathcal{C}\boxtimes\mathcal{D}}(X_1 \boxtimes X_2, Y_1 \boxtimes Y_2).$$

(v) any bilinear bifunctor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$ exact in each variable defines an exact functor $\bar{F} : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{A}$.

Proof. (sketch). (ii) follows from the universal property in the usual way.

(i) As we know, a finite abelian category is equivalent to the category of finite dimensional modules over an algebra. So there exist finite dimensional algebras C, D such that $\mathcal{C} = C - \text{mod}$, $\mathcal{D} = D - \text{mod}$. Then one can define $\mathcal{C} \boxtimes \mathcal{D} = C \otimes D - \text{mod}$, and it is easy to show that it satisfies the required conditions. This together with (ii) also implies (iii).

(iv),(v) are routine. □

A similar result is valid for locally finite categories.

Deligne's tensor product can also be applied to functors. Namely, if $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{D} \to \mathcal{D}'$ are additive right exact functors between finite abelian categories then one can define the functor $F \boxtimes G : \mathcal{C} \boxtimes D \to \mathcal{C}' \boxtimes \mathcal{D}'$.

Proposition 1.46.3. *If $\mathcal{C}, \mathcal{D}$ are multitensor categories then the category $\mathcal{C} \boxtimes \mathcal{D}$ has a natural structure of a multitensor category.*

Proof. Let $X_1 \boxtimes Y_1, X_2 \boxtimes Y_2 \in \mathcal{C} \boxtimes D$. Then we can set

$$(X_1 \boxtimes Y_1) \otimes (X_2 \boxtimes Y_2) := (X_1 \otimes X_2) \boxtimes (Y_1 \boxtimes Y_2).$$

and define the associativity isomorphism in the obvious way. This defines a structure of a monoidal category on the subcategory of $\mathcal{C} \boxtimes \mathcal{D}$ consisting of "$\boxtimes$-decomposable" objects of the form $X \boxtimes Y$. But any object of $\mathcal{C} \boxtimes D$ admits a resolution by $\boxtimes$-decomposable injective objects. This allows us to use a standard argument with resolutions to extend the tensor product to the entire category $\mathcal{C} \boxtimes D$. It is easy to see that if $\mathcal{C}$, $\mathcal{D}$ are rigid, then so is $\mathcal{C} \boxtimes D$, which implies the statement. □

1.47. **Finite (multi)tensor categories.** In this subsection we will study general properties of finite multitensor and tensor categories.

Recall that in a finite abelian category, every simple object X has a projective cover $P(X)$. The object $P(X)$ is unique up to a non-unique isomorphism. For any Y in $\mathcal{C}$ one has

$$\dim \mathsf{Hom}(P(X), Y) = [Y : X]. \tag{1.47.1}$$

Let $K_0(\mathcal{C})$ denote the free abelian group generated by isomorphism classes of indecomposable projective objects of a finite abelian category $\mathcal{C}$. Elements of $K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ will be called *virtual* projective objects. We have an obvious homomorphism $\gamma : K_0(\mathcal{C}) \to \mathsf{Gr}(\mathcal{C})$. Although groups $K_0(\mathcal{C})$ and $\mathsf{Gr}(\mathcal{C})$ have the same rank, in general γ is neither surjective nor injective even after tensoring with $\mathbb{C}$. The matrix C of γ in the natural basis is called the *Cartan matrix* of $\mathcal{C}$; its entries are $[P(X) : Y]$, where X, Y are simple objects of $\mathcal{C}$.

Now let $\mathcal{C}$ be a finite multitensor category, let I be the set of isomorphism classes of simple objects of $\mathcal{C}$, and let $i^*, {}^*i$ denote the right and left duals to i, respectively. Let $\mathsf{Gr}(\mathcal{C})$ be the Grothendieck ring of $\mathcal{C}$, spanned by isomorphism classes of the simple objects X_i, $i \in I$. In this ring, we have $X_i X_j = \sum_k N_{ij}^k X_k$, where N_{ij}^k are nonnegative integers. Also, let P_i denote the projective covers of X_i.

Proposition 1.47.1. *Let $\mathcal{C}$ be a finite multitensor category. Then $K_0(\mathcal{C})$ is a $\mathsf{Gr}(\mathcal{C})$-bimodule.*

Proof. This follows from the fact that the tensor product of a projective object with any object is projective, Proposition 1.13.6. □

Let us describe this bimodule explicitly.

Proposition 1.47.2. *For any object Z of $\mathcal{C}$,*

$$P_i \otimes Z \cong \oplus_{j,k} N^i_{kj^*}[Z : X_j]P_k, \;\; Z \otimes P_i \cong \oplus_{j,k} N^i_{*jk}[Z : X_j]P_k.$$

Proof. $\mathsf{Hom}(P_i \otimes Z, X_k) = \mathsf{Hom}(P_i, X_k \otimes Z^*)$, and the first formula follows from Proposition 1.13.6. The second formula is analogous. □

Proposition 1.47.3. *Let P be a projective object in a multitensor category $\mathcal{C}$. Then P^* is also projective. Hence, any projective object in a multitensor category is also injective.*

Proof. We need to show that the functor $\mathsf{Hom}(P^*, \bullet)$ is exact. This functor is isomorphic to $\mathsf{Hom}(\mathbf{1}, P \otimes \bullet)$. The functor $P \otimes \bullet$ is exact and moreover, by Proposition 1.13.6, any exact sequence splits after tensoring with P, as an exact sequence consisting of projective objects. The Proposition is proved. □

Proposition 1.47.3 implies that an indecomposable projective object P has a unique simple subobject, i.e. that the socle of P is simple.

For any finite tensor category $\mathcal{C}$ define an element $R_{\mathcal{C}} \in K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ by

$$R_{\mathcal{C}} = \sum_{i \in I} \mathsf{FPdim}(X_i) P_i. \tag{1.47.2}$$

Definition 1.47.4. The virtual projective object $R_{\mathcal{C}}$ is called the *regular object* of $\mathcal{C}$.

Definition 1.47.5. Let $\mathcal{C}$ be a finite tensor category. Then the *Frobenius-Perron* dimension of $\mathcal{C}$ is defined by

$$\mathsf{FPdim}(\mathcal{C}) := \mathsf{FPdim}(R_{\mathcal{C}}) = \sum_{i \in I} \mathsf{FPdim}(X_i)\, \mathsf{FPdim}(P_i). \tag{1.47.3}$$

Example 1.47.6. Let H be a finite dimensional quasi-Hopf algebra. Then $\mathsf{FPdim}(\mathrm{Rep}(H)) = \dim(H)$.

Proposition 1.47.7.
(1) *$Z \otimes R_{\mathcal{C}} = R_{\mathcal{C}} \otimes Z = \mathsf{FPdim}(Z) R_{\mathcal{C}}$ for all $Z \in \mathsf{Gr}(\mathcal{C})$.*
(2) *The image of $R_{\mathcal{C}}$ in $\mathsf{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ is a regular element.*

Proof. We have $\sum_i \mathsf{FPdim}(X_i)\dim \mathsf{Hom}(P_i, Z) = \mathsf{FPdim}(Z)$ for any object Z of $\mathcal{C}$. Hence,

$$\begin{aligned}
\sum_i \mathsf{FPdim}(X_i)\dim \mathsf{Hom}(P_i\otimes Z, Y) &= \sum_i \mathsf{FPdim}(X_i)\dim \mathsf{Hom}(P_i, Y\otimes Z^*)\\
&= \mathsf{FPdim}(Y\otimes Z^*)\\
&= \mathsf{FPdim}(Y)\,\mathsf{FPdim}(Z^*)\\
&= \mathsf{FPdim}(Y)\,\mathsf{FPdim}(Z)\\
&= \mathsf{FPdim}(Z)\sum_i \mathsf{FPdim}(X_i)\dim \mathsf{Hom}(P_i, Y).
\end{aligned}$$

Now, $P(X)\otimes Z$ are projective objects by Proposition 1.13.6. Hence, the formal sums $\sum_i \mathsf{FPdim}(X_i)P_i\otimes Z = R_{\mathcal{C}}\otimes Z$ and $\mathsf{FPdim}(Z)\sum_i \mathsf{FPdim}(X_i)P_i = \mathsf{FPdim}(Z)R_{\mathcal{C}}$ are linear combinations of P_j, $j\in I$ with the same coefficients. □

Remark 1.47.8. We note the following useful inequality:

$$\mathsf{FPdim}(\mathcal{C}) \geq N\,\mathsf{FPdim}(P), \tag{1.47.4}$$

where N is the number of simple objects in $\mathcal{C}$, and P is the projective cover of the neutral object $\mathbf{1}$. Indeed, for any simple object V the projective object $P(V)\otimes {}^*V$ has a nontrivial homomorphism to $\mathbf{1}$, and hence contains P. So $\mathsf{FPdim}(P(V))\,\mathsf{FPdim}(V) \geq \mathsf{FPdim}(P)$. Adding these inequalities over all simple V, we get the result.

1.48. **Integral tensor categories.**

Definition 1.48.1. A transitive unital $\mathbb{Z}_+$-ring A of finite rank is said to be integral if $\mathsf{FPdim} : A \to \mathbb{Z}$ (i.e. the Frobenius-Perron dimnensions of elements of $\mathcal{C}$ are integers). A tensor category $\mathcal{C}$ is integral if $\mathrm{Gr}(\mathcal{C})$ is integral.

Proposition 1.48.2. *A finite tensor category $\mathcal{C}$ is integral if and only if $\mathcal{C}$ is equivalent to the representation category of a finite dimensional quasi-Hopf algebra.*

Proof. The "if" part is clear from Example 1.45.6. To prove the "only if" part, it is enough to construct a quasi-fiber functor on $\mathcal{C}$. Define $P = \oplus_i \mathsf{FPdim}(X_i)P_i$, where X_i are the simple objects of $\mathcal{C}$, and P_i are their projective covers. Define $F = \mathsf{Hom}(P, \bullet)$. Obviously, F is exact and faithful, $F(\mathbf{1}) \cong \mathbf{1}$, and $\dim F(X) = \mathsf{FPdim}(X)$ for all $X \in \mathcal{C}$. Using Proposition 1.46.2, we continue the functors $F(\bullet\otimes\bullet)$ and $F(\bullet)\otimes F(\bullet)$ to the functors $\mathcal{C}\boxtimes\mathcal{C} \to \mathrm{Vec}$. Both of these functors are exact and take the same values on the simple objects of $\mathcal{C}\boxtimes\mathcal{C}$. Thus these functors are isomorphic and we are done. □

Corollary 1.48.3. *The assignment $H \mapsto \mathrm{Rep}(H)$ defines a bijection between integral finite tensor categories $\mathcal{C}$ over k up to monoidal equivalence, and finite dimensional quasi-Hopf algebras H over k, up to twist equivalence and isomorphism.*

1.49. **Surjective quasi-tensor functors.** Let $\mathcal{C}$, $\mathcal{D}$ be abelian categories. Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor.

Definition 1.49.1. We will say that F is *surjective* if any object of $\mathcal{D}$ is a subquotient in $F(X)$ for some $X \in \mathcal{C}$. [13]

Exercise 1.49.2. Let A, B be coalgebras, and $f : A \to B$ a homomorphism. Let $F = f^* : A - \mathrm{comod} \to B - \mathrm{comod}$ be the corresponding pushforward functor. Then F is surjective if and only if f is surjective.

Now let $\mathcal{C}$, $\mathcal{D}$ be finite tensor categories.

Theorem 1.49.3. *(*[EO]*) Let $F : \mathcal{C} \to \mathcal{D}$ be a surjective quasi-tensor functor. Then F maps projective objects to projective ones.*

Proof. Let $\mathcal{C}$ be a finite tensor category, and $X \in \mathcal{C}$. Let us write X as a direct sum of indecomposable objects (such a representation is unique). Define the *projectivity defect* $p(X)$ of X to be the sum of Frobenius-Perron dimensions of all the non-projective summands in this sum (this is well defined by the Krull-Schmidt theorem). It is clear that $p(X \oplus Y) = p(X) + p(Y)$. Also, it follows from Proposition 1.13.6 that $p(X \otimes Y) \le p(X)p(Y)$.

Let P_i be the indecomposable projective objects in $\mathcal{C}$. Let $P_i \otimes P_j \cong \oplus_k B_{ij}^k P_k$, and let B_i be the matrix with entries B_{ij}^k. Also, let $B = \sum B_i$. Obviously, B has strictly positive entries, and the Frobenius-Perron eigenvalue of B is $\sum_i \mathsf{FPdim}(P_i)$.

On the other hand, let $F : \mathcal{C} \to \mathcal{D}$ be a surjective quasi-tensor functor between finite tensor categories. Let $p_j = p(F(P_j))$, and $\mathbf{p}$ be the vector with entries p_j. Then we get $p_i p_j \ge \sum_k B_{ij}^k p_k$, so $(\sum_i p_i)\mathbf{p} \ge B\mathbf{p}$. So, either p_i are all zero, or they are all positive, and the norm of B with respect to the norm $|x| = \sum p_i |x_i|$ is at most $\sum p_i$. Since $p_i \le \mathsf{FPdim}(P_i)$, this implies $p_i = \mathsf{FPdim}(P_i)$ for all i (as the largest eigenvalue of B is $\sum_i \mathsf{FPdim}(P_i)$).

Assume the second option is the case. Then $F(P_i)$ do not contain nonzero projective objects as direct summands, and hence for any projective $P \in \mathcal{C}$, $F(P)$ cannot contain a nonzero projective object as a direct summand. However, let Q be a projective object of $\mathcal{D}$. Then,

[13] This definition does not coincide with a usual categorical definition of surjectivity of functors which requires that every object of $\mathcal{D}$ be isomorphic to some $F(X)$ for an object X in $\mathcal{C}$.

since F is surjective, there exists an object $X \in \mathcal{C}$ such that Q is a subquotient of $F(X)$. Since any X is a quotient of a projective object, and F is exact, we may assume that $X = P$ is projective. So Q occurs as a subquotient in $F(P)$. As Q is both projective and injective, it is actually a direct summand in $F(P)$. Contradiction.

Thus, $p_i = 0$ and $F(P_i)$ are projective. The theorem is proved. $\square$

1.50. **Categorical freeness.** Let $\mathcal{C}, \mathcal{D}$ be finite tensor categories, and $F : \mathcal{C} \to \mathcal{D}$ be a quasi-tensor functor.

Theorem 1.50.1. *One has*

$$F(R_{\mathcal{C}}) = \frac{\mathsf{FPdim}(\mathcal{C})}{\mathsf{FPdim}(\mathcal{D})} R_{\mathcal{D}}. \tag{1.50.1}$$

Proof. By Theorem 1.49.3, $F(R_{\mathcal{C}})$ is a virtually projective object. Thus, $F(R_{\mathcal{C}})$ must be proportional to $R_{\mathcal{D}}$, since both (when written in the basis P_i) are eigenvectors of a matrix with strictly positive entries with its Frobenius-Perron eigenvalue. (For this matrix we may take the matrix of multiplication by $F(X)$, where X is such that $F(X)$ contains as composition factors all simple objects of $\mathcal{D}$; such exists by the surjectivity of F). The coefficient is obtained by computing the Frobenius-Perron dimensions of both sides. $\square$

Corollary 1.50.2. *In the above situation, one has* $\mathsf{FPdim}(\mathcal{C}) \geq \mathsf{FPdim}(\mathcal{D})$, *and* $\mathsf{FPdim}(\mathcal{D})$ *divides* $\mathsf{FPdim}(\mathcal{C})$ *in the ring of algebraic integers. In fact,*

$$\frac{\mathsf{FPdim}(\mathcal{C})}{\mathsf{FPdim}(\mathcal{D})} = \sum \mathsf{FPdim}(X_i) \dim \mathsf{Hom}(F(P_i), \mathbf{1}_{\mathcal{D}}), \tag{1.50.2}$$

where X_i runs over simple objects of $\mathcal{C}$.

Proof. The statement is obtained by computing the dimension of $\mathsf{Hom}(\bullet, \mathbf{1}_{\mathcal{D}})$ for both sides of (1.50.1). $\square$

Suppose now that $\mathcal{C}$ is integral, i.e., by Proposition 1.48.2, it is the representation category of a quasi-Hopf algebra H. In this case, $R_{\mathcal{C}}$ is an honest (not only virtual) projective object of $\mathcal{C}$, namely the free rank 1 module over H. Theorefore, multiples of $R_{\mathcal{C}}$ are free H-modules of finite rank, and vice versa.

Then Theorem 1.49.3 and the fact that $F(R_{\mathcal{C}})$ is proportional to $R_{\mathcal{D}}$ implies the following categorical freeness result.

Corollary 1.50.3. *If $\mathcal{C}$ is integral, and $F : \mathcal{C} \to \mathcal{D}$ is a surjective quasi-tensor functor then $\mathcal{D}$ is also integral, and the object $F(R_{\mathcal{C}})$ is free of rank* $\mathsf{FPdim}(\mathcal{C}) / \mathsf{FPdim}(\mathcal{D})$ *(which is an integer).*

Proof. The Frobenius-Perron dimensions of simple objects of $\mathcal{D}$ are coordinates of the unique eigenvector of the positive integer matrix of multiplication by $F(R_{\mathcal{C}})$ with integer eigenvalue $\mathsf{FPdim}(\mathcal{C})$, normalized so that the component of $\mathbf{1}$ is 1. Thus, all coordinates of this vector are rational numbers, hence integers (because they are algebraic integers). This implies that the category $\mathcal{D}$ is integral. The second statement is clear from the above. $\square$

Corollary 1.50.4. *(*[Scha]*; for the semisimple case see* [ENO1]*) A finite dimensional quasi-Hopf algebra is a free module over its quasi-Hopf subalgebra.*

Remark 1.50.5. In the Hopf case Corollary 1.50.3 is well known and much used; it is due to Nichols and Zoeller [NZ].

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