2.7. **First properties of exact module categories.**

Lemma 2.7.1. *Let $\mathcal{M}$ be an exact module category over finite multitensor category $\mathcal{C}$. Then the category $\mathcal{M}$ has enough projective objects.*

Proof. Let P_0 denote the projective cover of the unit object in $\mathcal{C}$. Then the natural map $P_0 \otimes X \to \mathbf{1} \otimes X \simeq X$ is surjective for any $X \in \mathcal{M}$ since $\otimes$ is exact. Also $P_0 \otimes X$ is projective by definition of an exact module category. □

Corollary 2.7.2. *Assume that an exact module category $\mathcal{M}$ over $\mathcal{C}$ has finitely many isomorphism classes of simple objects. Then $\mathcal{M}$ is finite.*

Lemma 2.7.3. *Let $\mathcal{M}$ be an exact module category over $\mathcal{C}$. Let $P \in \mathcal{C}$ be projective and $X \in \mathcal{M}$. Then $P \otimes X$ is injective.*

Proof. The functor $\mathsf{Hom}(\bullet, P \otimes X)$ is isomorphic to the functor $\mathsf{Hom}(P^* \otimes \bullet, X)$. The object P^* is projective by Proposition 1.47.3. Thus for any exact sequence

$$0 \to Y_1 \to Y_2 \to Y_3 \to 0$$

the sequence

$$0 \to P^* \otimes Y_1 \to P^* \otimes Y_2 \to P^* \otimes Y_3 \to 0$$

splits, and hence the functor $\mathsf{Hom}(P^* \otimes \bullet, X)$ is exact. The Lemma is proved. □

Corollary 2.7.4. *In the category $\mathcal{M}$ any projective object is injective and vice versa.*

Proof. Any projective object X of $\mathcal{M}$ is a direct summand of the object of the form $P_0 \otimes X$ and thus is injective. □

Remark 2.7.5. A finite abelian category $\mathcal{A}$ is called a quasi-Frobenius category if any projective object of $\mathcal{A}$ is injective and vice versa. Thus any exact module category over a finite multitensor category (in particular, any finite multitensor category itself) is a quasi-Frobenius category. It is well known that any object of a quasi-Frobenius category admitting a finite projective resolution is projective (indeed, the last nonzero arrow of this resolution is an embedding of projective (= injective) modules and therefore is an inclusion of a direct summand. Hence the resolution can be replaced by a shorter one and by induction we are done). Thus any quasi-Frobenius category is either semisimple or of infinite homological dimension.

Let $Irr(\mathcal{M})$ denote the set of (isomorphism classes of) simple objects in $\mathcal{M}$. Let us introduce the following relation on $Irr(\mathcal{M})$: two objects $X, Y \in Irr(\mathcal{M})$ are related if Y appears as a subquotient of $L \otimes X$ for some $L \in \mathcal{C}$.

Lemma 2.7.6. *The relation above is reflexive, symmetric and transitive.*

Proof. Since $\mathbf{1} \otimes X = X$ we have the reflexivity. Let $X, Y, Z \in Irr(\mathcal{M})$ and $L_1, L_2 \in \mathcal{C}$. If Y is a subquotient of $L_1 \otimes X$ and Z is a subquotient of $L_2 \otimes Y$ then Z is a subquotient of $(L_2 \otimes L_1) \otimes X$ (since $\otimes$ is exact), so we get the transitivity. Now assume that Y is a subquotient of $L \otimes X$. Then the projective cover $P(Y)$ of Y is a direct summand of $P_0 \otimes L \otimes X$; hence there exists $S \in \mathcal{C}$ such that $\mathsf{Hom}(S \otimes X, Y) \neq 0$ (for example $S = P_0 \otimes L$). Thus $\mathsf{Hom}(X, S^* \otimes Y) = \mathsf{Hom}(S \otimes X, Y) \neq 0$ and hence X is a subobject of $S^* \otimes Y$. Consequently our equivalence relation is symmetric. $\square$

Thus our relation is an equivalence relation. Hence $Irr(\mathcal{M})$ is partitioned into equivalence classes, $Irr(\mathcal{M}) = \bigsqcup_{i \in I} Irr(\mathcal{M})_i$. For an equivalence class $i \in I$ let $\mathcal{M}_i$ denote the full subcategory of $\mathcal{M}$ consisting of objects whose all simple subquotients lie in $Irr(\mathcal{M})_i$. Clearly, $\mathcal{M}_i$ is a module subcategory of $\mathcal{M}$.

Proposition 2.7.7. *The module categories $\mathcal{M}_i$ are exact. The category $\mathcal{M}$ is the direct sum of its module subcategories $\mathcal{M}_i$.*

Proof. For any $X \in Irr(\mathcal{M})_i$ its projective cover is a direct summand of $P_0 \otimes X$ and hence lies in the category $\mathcal{M}_i$. Hence the category $\mathcal{M}$ is the direct sum of its subcategories $\mathcal{M}_i$, and $\mathcal{M}_i$ are exact. $\square$

A crucial property of exact module categories is the following

Proposition 2.7.8. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two module categories over $\mathcal{C}$. Assume that $\mathcal{M}_1$ is exact. Then any additive module functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ is exact.*

Proof. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathcal{M}_1$. Assume that the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is not exact. Then the sequence $0 \to P \otimes F(X) \to P \otimes F(Y) \to P \otimes F(Z) \to 0$ is also non-exact for any nonzero object $P \in \mathcal{C}$ since the functor $P \otimes \bullet$ is exact and $P \otimes X = 0$ implies $X = 0$. In particular we can take P to be projective. But then the sequence $0 \to P \otimes X \to P \otimes Y \to P \otimes Z \to 0$ is exact and split and hence the sequence $0 \to F(P \otimes X) \to F(P \otimes Y) \to F(P \otimes Z) \to 0$ is exact and we get a contradiction. $\square$

Remark 2.7.9. We will see later that this Proposition actually characterizes exact module categories.

2.8. $\mathbb{Z}_+-$**modules.** Recall that for any multitensor category $\mathcal{C}$ its Grothendieck ring $Gr(\mathcal{C})$ is naturally a $\mathbb{Z}_+-$ring.

Definition 2.8.1. Let K be a $\mathbb{Z}_+-$ring with basis $\{b_i\}$. A $\mathbb{Z}_+-$*module* over K is a $K-$module M with fixed $\mathbb{Z}-$basis $\{m_l\}$ such that all the structure constants a_{il}^k (defined by the equality $b_i m_l = \sum_k a_{il}^k m_k$) are nonnegative integers.

The direct sum of $\mathbb{Z}_+-$modules is also a $\mathbb{Z}_+-$module whose basis is a union of bases of summands. We say that $\mathbb{Z}_+-$module is *indecomposable* if it is not isomorphic to a nontrivial direct sum.

Let $\mathcal{M}$ be a finite module category over $\mathcal{C}$. By definition, the Grothendieck group $Gr(\mathcal{M})$ with the basis given by the isomorphism classes of simple objects is a $\mathbb{Z}_+-$module over $Gr(\mathcal{C})$. Obviously, the direct sum of module categories corresponds to the direct sum of $\mathbb{Z}_+-$modules.

Exercise 2.8.2. Construct an example of an indecomposable module category $\mathcal{M}$ over $\mathcal{C}$ such that $Gr(\mathcal{M})$ is not indecomposable over $Gr(\mathcal{C})$.

Note, however, that, as follows immediately from Proposition 2.7.7, for an indecomposable exact module category $\mathcal{M}$ the $\mathbb{Z}_+-$module $Gr(\mathcal{M})$ is indecomposable over $Gr(\mathcal{C})$. In fact, even more is true.

Definition 2.8.3. A $\mathbb{Z}_+-$module M over a $\mathbb{Z}_+-$ring K is called *irreducible* if it has no proper $\mathbb{Z}_+-$submodules (in other words, the $\mathbb{Z}-$span of any proper subset of the basis of M is not a $K-$submodule).

Exercise 2.8.4. Give an example of $\mathbb{Z}_+-$module which is not irreducible but is indecomposable.

Lemma 2.8.5. *Let $\mathcal{M}$ be an indecomposable exact module category over $\mathcal{C}$. Then $Gr(\mathcal{M})$ is an irreducible $\mathbb{Z}_+-$module over $Gr(\mathcal{C})$.*

Exercise 2.8.6. Prove this Lemma.

Proposition 2.8.7. *Let K be a based ring of finite rank over $\mathbb{Z}$. Then there exists only finitely many irreducible $\mathbb{Z}_+-$modules over K.*

Proof. First of all, it is clear that an irreducible $\mathbb{Z}_+-$module M over K is of finite rank over $\mathbb{Z}$. Let $\{m_l\}_{l\in L}$ be the basis of M. Let us consider an element $b := \sum_{b_i\in B} b_i$ of K. Let $b^2 = \sum_i n_i b_i$ and let $N = \max_{b_i\in B} n_i$ (N exists since B is finite). For any $l \in L$ let $bm_l = \sum_{k\in L} d_l^k m_k$ and let $d_l := \sum_{k\in L} d_l^k > 0$. Let $l_0 \in I$ be such that $d := d_{l_0}$ equals $\min_{l\in L} d_l$. Let $b^2 m_{l_0} = \sum_{l\in L} c_l m_l$. Calculating $b^2 m_{l_0}$ in two ways

— as $(b^2)m_{l_0}$ and as $b(bm_{l_0})$, and computing the sum of the coefficients, we have:

$$Nd \geq \sum_l c_l \geq d^2$$

and consequently $d \leq N$. So there are only finitely many possibilities for $|L|$, values of c_i and consequently for expansions $b_i m_l$ (since each m_l appears in bm_{l_0}). The Proposition is proved.

□

In particular, for a given finite multitensor category $\mathcal{C}$ there are only finitely many $\mathbb{Z}_+-$modules over $Gr(\mathcal{C})$ which are of the form $Gr(\mathcal{M})$ where $\mathcal{M}$ is an indecomposable exact module category over $\mathcal{C}$.

Exercise 2.8.8. (a) Classify irreducible $\mathbb{Z}_+-$modules over $\mathbb{Z}G$ (Answer: such modules are in bijection with subgroups of G up to conjugacy).

(b) Classify irreducible $\mathbb{Z}_+-$modules over $Gr(\mathsf{Rep}(S_3))$ (consider all the cases: $chark \neq 2, 3$, $chark = 2$, $chark = 3$).

(c) Classify irreducible $\mathbb{Z}_+-$modules over the Yang-Lee and Ising based rings.

Now we can suggest an approach to the classification of exact module categories over $\mathcal{C}$: first classify irreducible $\mathbb{Z}_+-$modules over $Gr(\mathcal{C})$ (this is a combinatorial part), and then try to find all possible categorifications of a given $\mathbb{Z}_+-$module (this is a categorical part). Both these problems are quite nontrivial and interesting. We will see later some nontrivial solutions to this.

2.9. **Algebras in categories.**

Definition 2.9.1. An algebra in a multitensor category $\mathcal{C}$ is a triple (A, m, u) where A is an object of $\mathcal{C}$, and m, u are morphisms (called multiplication and unit morphisms) $m : A \otimes A \to A$, $u : \mathbf{1} \to A$ such that the following axioms are satisfied:

1. **Associativity**: the following diagram commutes:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes id} & A \otimes A \\ {\scriptstyle id \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow[m]{} & A \end{array} \qquad (2.9.1)$$

2. **Unit**: The morphisms $A \to \mathbf{1} \otimes A \to A \otimes A \to A$ and $A \to A \otimes \mathbf{1} \to A \otimes A \to A$ are both equal to Id_A.

Of course, in the case when $\mathcal{C} = \mathbf{Vec}$, we get definition of an associative algebra with unit, and in the case $\mathcal{C} = \mathrm{Vec}$ we get the definition of a finite dimensional associative algebra with unit.

Remark 2.9.2. If $\mathcal{C}$ is not closed under direct limits (e.g., $\mathcal{C}$ is a multitensor category), one can generalize the above definition, allowing A to be an ind-object (i.e., "infinite dimensional"). However, we will mostly deal with algebras honestly in $\mathcal{C}$ (i.e., "finite dimensional"), and will make this assumption unless otherwise specified.

Example 2.9.3. 1. $\mathbf{1}$ is an algebra.

2. The algebra of functions $\mathrm{Fun}(G)$ on a finite group G (with values in the ground field k) is an algebra in $\mathrm{Rep}(G)$ (where G acts on itself by left multiplication).

3. Algebras in Vec_G is the same thing as G-graded algebras. In particular, if H is a subgroup of G then the group algebra $\mathbb{C}[H]$ is an algebra in Vec_G.

4. More generally, let ω be a 3-cocycle on G with values in $k^\times$, and ψ be a 2-cochain of G such that $\omega = d\psi$. Then one can define the *twisted group algebra* $\mathbb{C}_\psi[H]$ in Vec_G^ω, which is $\oplus_{h\in H} h$ as an object of Vec_G^ω, and the multiplication $h \otimes h' \to hh'$ is the operation of multiplication by $\psi(h, h')$. If $\omega = 1$ (i.e., ψ is a 2-cocycle), the twisted group algebra is associative in the usual sense, and is a familiar object from group theory. However, if ω is nontrivial, this algebra is not associative in the usual sense, but is only associative in the tensor category Vec_G^ω, which, as we know, does not admit fiber functors.

Example 2.9.4. Let $\mathcal{C}$ be a multitensor category and $X \in \mathcal{C}$. Then the object $A = X \otimes X^*$ has a natural structure of an algebra with unit in $\mathcal{C}$ given by the coevaluation morphism and multiplication $\mathsf{Id} \otimes \mathrm{ev}_X \otimes \mathsf{Id}$. In particular for $X = \mathbf{1}$ we get a (trivial) structure of an algebra on $A = \mathbf{1}$.

We leave it to the reader to define subalgebras, homomorphisms, ideals etc in the categorical setting.

Now we define modules over algebras:

Definition 2.9.5. A (right) module over an algebra (A, m, u) (or just an $A-$module) is a pair (M, p), where $M \in \mathcal{C}$ and p is a morphism $M \otimes A \to M$ such that the following axioms are satisfied:

1. The following diagram commutes:

$$
\begin{array}{ccc}
M \otimes A \otimes A & \xrightarrow{p\otimes id} & M \otimes A \\
{\scriptstyle id\otimes m}\downarrow & & \downarrow{\scriptstyle p} \\
M \otimes A & \xrightarrow[p]{} & M
\end{array} \tag{2.9.2}
$$

2. The composition $M \to M \otimes \mathbf{1} \to M \otimes A \to M$ is the identity.

The definition of a left module is entirely analogous.

Definition 2.9.6. The homomorphism between two $A-$modules (M_1, p_1) and (M_2, p_2) is a morphism $l \in \mathsf{Hom}_{\mathcal{C}}(M_1, M_2)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
M_1 \otimes A & \xrightarrow{l\otimes id} & M_2 \otimes A \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\
M_1 & \xrightarrow[l]{} & M_2
\end{array} \tag{2.9.3}
$$

Obviously, homomorphisms form a subspace of the the vector space $\mathsf{Hom}(M_1, M_2)$. We will denote this subspace by $\mathsf{Hom}_A(M_1, M_2)$. It is easy to see that a composition of homomorphisms is a homomorphism. Thus $A-$modules form a category $Mod_{\mathcal{C}}(A)$.

Exercise 2.9.7. Check that $Mod_{\mathcal{C}}(A)$ is an abelian category.

The following observations relate the categories $Mod_{\mathcal{C}}(A)$ and module categories:

Exercise 2.9.8. For any $A-$module (M, p) and any $X \in \mathcal{C}$ the pair $(X \otimes M, id \otimes p)$ is again an $A-$module.

Thus we have a functor $\tilde{\otimes} : \mathcal{C} \times Mod_{\mathcal{C}}(A) \to Mod_{\mathcal{C}}(A)$.

Exercise 2.9.9. For any $A-$module (M, p) and any $X, Y \in \mathcal{C}$ the associativity morphism $a_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$ is an isomorphism of $A-$modules. Similarly the unit morphism $\mathbf{1} \otimes M \to M$ is an isomorphism of $A-$modules.

This exercise defines associativity and unit constraints $\tilde{a}, \tilde{l}$ for the category $Mod_{\mathcal{C}}(A)$.

Proposition 2.9.10. *The category $Mod_{\mathcal{C}}(A)$ together with functor $\tilde{\otimes}$ and associativity and unit constraints $\tilde{a}$, $\tilde{l}$ is a left module category over $\mathcal{C}$.*

Exercise 2.9.11. Prove this Proposition.

The following statement is very useful:

Lemma 2.9.12. *For any $X \in \mathcal{C}$ we have a canonical isomorphism* $\mathsf{Hom}_A(X \otimes A, M) = \mathsf{Hom}(X, M)$*.*

Exercise 2.9.13. Prove this Lemma.

Exercise 2.9.14. Is it true that any object of $Mod_{\mathcal{C}}(A)$ is of the form $X \otimes A$ for some $X \in \mathcal{C}$?

Exercise 2.9.15. Show that for any $M \in Mod_{\mathcal{C}}(A)$ there exists $X \in \mathcal{C}$ and a surjection $X \otimes A \to M$ (namely, $X = M$ regarded as an object of $\mathcal{C}$).

Exercise 2.9.16. Assume that the category $\mathcal{C}$ has enough projective objects. Then the category $Mod_{\mathcal{C}}(A)$ has enough projective objects.

Exercise 2.9.17. Assume that the category $\mathcal{C}$ is finite. Then the category $Mod_{\mathcal{C}}(A)$ is finite.

Thus we get a general construction of module categories from algebras in the category $\mathcal{C}$. Not any module category over $\mathcal{C}$ is of the form $Mod_{\mathcal{C}}(A)$: for $\mathcal{C} = \mathrm{Vec}$ the module category of all (possibly infinite dimensional) vector spaces (see Example 2.5.11) is not of this form. But note that for $\mathcal{C} = \mathrm{Vec}$ any finite module category is of the form $Mod_{\mathcal{C}}(A)$ (just because every finite abelian category is equivalent to $Mod(A)$ for some finite dimensional algebra A). We will show later that all finite module categories over a finite $\mathcal{C}$ are of the form $Mod_{\mathcal{C}}(A)$ for a suitable A. But of course different algebras A can give rise to the same module categories.

Definition 2.9.18. We say that two algebras A and B in $\mathcal{C}$ are *Morita equivalent* if the module categories $Mod_{\mathcal{C}}(A)$ and $Mod_{\mathcal{C}}(B)$ are module equivalent.

Note that in the case $\mathcal{C} = \mathrm{Vec}$ this definition specializes to the usual notion of Morita equivalence of finite dimensional algebras.

Example 2.9.19. We will see later that all the algebras from Example 2.9.4 are Morita equivalent; moreover any algebra which is Morita equivalent to $A = \mathbf{1}$ is of the form $X \otimes X^*$ for a suitable $X \in \mathcal{C}$.

Not any module category of the form $Mod_{\mathcal{C}}(A)$ is exact:

Exercise 2.9.20. Give an example of module category of the form $Mod_{\mathcal{C}}(A)$ which is not exact.

Thus we are going to use the following

Definition 2.9.21. An algebra A in the category $\mathcal{C}$ is called *exact* if the module category $Mod_{\mathcal{C}}(A)$ is exact.

It is obvious from the definition that the exactness is invariant under Morita equivalence.

We will need the notion of a tensor product over an algebra $A \in \mathcal{C}$.

Definition 2.9.22. Let A be an algebra in $\mathcal{C}$ and let (M, p_M) be a right $A-$module, and (N, p_N) be a left $A-$module. A *tensor product* over A, $M \otimes_A N \in \mathcal{C}$, is the quotient of $M \otimes N$ by the image of morphism $p_M \otimes id - id \otimes p_N : M \otimes A \otimes N \to M \otimes N$.

Exercise 2.9.23. Show that the functor $\otimes_A$ is right exact in each variable (that is, for fixed M, N, the functors $M \otimes_A \bullet$ and $\bullet \otimes_A N$ are right exact).

Definition 2.9.24. Let A, B be two algebras in $\mathcal{C}$. An $A-B-$*bimodule* is a triple (M, p, q) where $M \in \mathcal{C}$, $p \in \mathsf{Hom}(A \otimes M, M)$, $q \in \mathsf{Hom}(M \otimes B, M)$ such that

1. The pair (M, p) is a left $A-$module.
2. The pair (M, q) is a right $B-$module.
3. The morphisms $q \circ (p \otimes id)$ and $p \circ (id \otimes q)$ from $\mathsf{Hom}(A \otimes M \otimes B, M)$ coincide.

Remark 2.9.25. Note that in the categorical setting, we cannot define (A, B)-bimodules as modules over $A \otimes B^{op}$, since the algebra $A \otimes B^{op}$ is, in general, not defined.

We will usually say "$A-$bimodule" instead of "$A - A-$bimodule".

Exercise 2.9.26. Let M be a right $A-$module, N be an $A-B-$bimodule and P be a left $B-$module. Construct the associativity morphism $(M \otimes_A N) \otimes_A P \to M \otimes_A (N \otimes_A P)$. State and prove the pentagon relation for this morphism.

2.10. **Internal Hom.** In this section we assume that the category $\mathcal{C}$ is finite. This is not strictly necessary but simplifies the exposition.

An important technical tool in the study of module categories is the notion of internal Hom. Let $\mathcal{M}$ be a module category over $\mathcal{C}$ and $M_1, M_2 \in \mathcal{M}$. Consider the functor $\mathsf{Hom}(\bullet \otimes M_1, M_2)$ from the category $\mathcal{C}$ to the category of vector spaces. This functor is left exact and thus is representable

Remark 2.10.1. If we do not assume that the category $\mathcal{C}$ is finite, the functor above is still representable, but by an ind-object of $\mathcal{C}$. Working with ind-objects, one can extend the theory below to this more general case. We leave this for an interested reader.

Definition 2.10.2. The internal Hom $\underline{\mathrm{Hom}}(M_1, M_2)$ is an object of $\mathcal{C}$ representing the functor $\mathsf{Hom}(\bullet \otimes M_1, M_2)$.

Note that by Yoneda's Lemma $(M_1, M_2) \mapsto \underline{\mathrm{Hom}}(M_1, M_2)$ is a bifunctor.

Exercise 2.10.3. Show that the functor $\underline{\mathrm{Hom}}(\bullet, \bullet)$ is left exact in both variables.

Lemma 2.10.4. *There are canonical isomorphims*

$$\begin{array}{ll} (1) & \mathsf{Hom}(X \otimes M_1, M_2) \cong \mathsf{Hom}(X, \underline{Hom}(M_1, M_2)), \\ (2) & \mathsf{Hom}(M_1, X \otimes M_2) \cong \mathsf{Hom}(\mathbf{1}, X \otimes \underline{Hom}(M_1, M_2)), \\ (3) & \underline{Hom}(X \otimes M_1, M_2) \cong \underline{Hom}(M_1, M_2) \otimes X^*, \\ (4) & \underline{Hom}(M_1, X \otimes M_2) \cong X \otimes \underline{Hom}(M_1, M_2). \end{array}$$

Proof. Formula (1) is just the definition of $\underline{\mathrm{Hom}}(M_1, M_2)$, and isomorphism (2) is the composition

$$\mathsf{Hom}(M_1, X \otimes M_2) \cong \mathsf{Hom}(X^* \otimes M_1, M_2) =$$
$$= \mathsf{Hom}(X^*, \underline{\mathrm{Hom}}(M_1, M_2)) \cong \mathsf{Hom}(\mathbf{1}, X \otimes \underline{\mathrm{Hom}}(M_1, M_2)).$$

We get isomorphism (3) from the calculation

$$\mathsf{Hom}(Y, \underline{\mathrm{Hom}}(X \otimes M_1, M_2)) = \mathsf{Hom}(Y \otimes (X \otimes M_1), M_2) = \mathsf{Hom}((Y \otimes X) \otimes M_1, M_2) =$$
$$= \mathsf{Hom}(Y \otimes X, \underline{\mathrm{Hom}}(M_1, M_2)) = \mathsf{Hom}(Y, \underline{\mathrm{Hom}}(M_1, M_2) \otimes X^*),$$

and isomorphism (4) from the calculation

$$\mathsf{Hom}(Y, \underline{\mathrm{Hom}}(M_1, X \otimes M_2)) = \mathsf{Hom}(Y \otimes M_1, X \otimes M_2) =$$
$$= \mathsf{Hom}(X^* \otimes (Y \otimes M_1), M_2) = \mathsf{Hom}((X^* \otimes Y) \otimes M_1, M_2) =$$
$$= \mathsf{Hom}(X^* \otimes Y, \underline{\mathrm{Hom}}(M_1, M_2)) = \mathsf{Hom}(Y, X \otimes \underline{\mathrm{Hom}}(M_1, M_2)).$$

□

Corollary 2.10.5. *(1) For a fixed M_1, the assignment $M_2 \mapsto \underline{Hom}(M_1, M_2)$ is a module functor $\mathcal{M} \to \mathcal{C}$;*

(2) For a fixed M_2, the assignment $M_1 \mapsto \underline{Hom}(M_1, M_2)$ is a module functor $\mathcal{M} \to \mathcal{C}^{\mathrm{op}}$.

Proof. This follows from the isomorphisms (4) and (3) of Lemma 2.10.4. □

Corollary 2.10.5 and Proposition 2.7.8 imply

Corollary 2.10.6. *Assume that $\mathcal{M}$ is an exact module category. Then the functor $\underline{Hom}(\bullet, \bullet)$ is exact in each variable.*

The mere definition of the internal Hom allows us to prove the converse to Proposition 2.7.8:

Proposition 2.10.7. *(1) Suppose that for a module category $\mathcal{M}$ over $\mathcal{C}$, the bifunctor $\underline{Hom}$ is exact in the second variable, i.e., for any object $N \in \mathcal{M}$ the functor $\underline{Hom}(N, \bullet) : \mathcal{M} \to \mathcal{C}$ is exact. Then $\mathcal{M}$ is exact.*

(2) Let $\mathcal{M}_1, \mathcal{M}_2$ be two nonzero module categories over $\mathcal{C}$. Assume that any module functor from $\mathcal{M}_1$ to $\mathcal{M}_2$ is exact. Then the module category $\mathcal{M}_1$ is exact.

Proof. (1) Let $P \in \mathcal{C}$ be any projective object. Then for any $N \in \mathcal{M}$ one has $\mathsf{Hom}(P \otimes N, \bullet) = \mathsf{Hom}(P, \underline{\mathrm{Hom}}(N, \bullet))$, and thus the functor $\mathsf{Hom}(P \otimes N, \bullet)$ is exact. By the definition of an exact module category, we are done.

(2) First we claim that under our assumptions any module functor $F \in \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{C})$ is exact. Indeed, let $0 \neq M \in \mathcal{M}_2$. The functor $F(\bullet) \otimes M \in \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is exact. Since $\bullet \otimes M$ is exact, and $X \otimes M = 0$ implies $X = 0$, we see that F is exact.

In particular, we see that for any object $N \in \mathcal{M}_1$, the functor $\underline{\mathrm{Hom}}(N, \bullet) : \mathcal{M}_1 \to \mathcal{C}$ is exact, since it is a module functor. Now (2) follows from (1). □

Example 2.10.8. It is instructive to calculate $\underline{\mathrm{Hom}}$ for the category $Mod_{\mathcal{C}}(A)$. Let $M, N \in Mod_{\mathcal{C}}(A)$. We leave it to the reader as an exercise to check that $\underline{\mathrm{Hom}}(M, N) = (M \otimes_A {}^*N)^*$ (note that ${}^*N$ has a natural structure of a left $A-$module). One deduces from this description of $\underline{\mathrm{Hom}}$ that exactness of A is equivalent to biexactness of the functor $\otimes_A$.

For two objects M_1, M_2 of a module category $\mathcal{M}$ we have the canonical morphism

$$ev_{M_1,M_2} : \underline{\mathrm{Hom}}(M_1, M_2) \otimes M_1 \to M_2$$

obtained as the image of Id under the isomorphism

$$\mathsf{Hom}(\underline{\mathrm{Hom}}(M_1, M_2), \underline{\mathrm{Hom}}(M_1, M_2)) \cong \mathsf{Hom}(\underline{\mathrm{Hom}}(M_1, M_2) \otimes M_1, M_2).$$

Let M_1, M_2, M_3 be three objects of $\mathcal{M}$. Then there is a canonical composition morphism

$$(\underline{\mathrm{Hom}}(M_2, M_3) \otimes \underline{\mathrm{Hom}}(M_1, M_2)) \otimes M_1 \cong \underline{\mathrm{Hom}}(M_2, M_3) \otimes (\underline{\mathrm{Hom}}(M_1, M_2) \otimes M_1)$$

$$\stackrel{\mathsf{Id} \otimes ev_{M_1,M_2}}{\longrightarrow} \underline{\mathrm{Hom}}(M_2, M_3) \otimes M_2 \stackrel{ev_{M_2,M_3}}{\longrightarrow} M_3$$

which produces the *multipication morphism*

$$\underline{\mathrm{Hom}}(M_2, M_3) \otimes \underline{\mathrm{Hom}}(M_1, M_2) \to \underline{\mathrm{Hom}}(M_1, M_3).$$

Exercise 2.10.9. Check that this multiplication is associative and compatible with the isomorphisms of Lemma 2.10.4.

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