

18.781 Problem Set 6

Thursday, April 12.

Collaboration is allowed and encouraged. However, your writeups should be your own, and you must note on the front the names of the students you worked with.

Extensions will only be given for extenuating circumstances.

1. Show that $\prod_{d|n} d = n^{d(n)/2}$.
2. If k is a positive integer, show that $\sigma_k(n)$ is odd if and only if n is a square or twice a square.
3. Prove that if $(a, b) > 1$ then $\sigma_k(ab) < \sigma_k(a)\sigma_k(b)$ and $d(ab) < d(a)d(b)$.
4. Recall that a *perfect number* n is one for which $\sigma(n) = 2n$.
 - (a) If $p = 2^m - 1$ is a prime, show that $2^{m-1}(2^m - 1)$ is a perfect number.
 - (b) Show that every even perfect number is of the above form. [Hint: start by decomposing $n = 2^r s$ where s is odd.]
5. For any positive integer n , let $\lambda(n) = (-1)^{\Omega(n)}$. This is *Liouville's lambda function*. Show that $\lambda(n)$ is totally multiplicative and that

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 & n \text{ is a perfect square} \\ 0 & \text{otherwise.} \end{cases}$$

6. Show that

$$\sum_{d|n} d(d)^3 = \left(\sum_{d|n} d(d) \right)^2$$

for all positive integers n .

7. (a) Suppose $f(n)$ is an arithmetic function whose values are all nonzero, and put $\widehat{F}(n) = \prod_{d|n} f(d)$. Show that

$$f(n) = \prod_{d|n} \widehat{F}(d)^{\mu(n/d)}.$$

- (b) Show that

$$\prod_{\substack{a=1 \\ (a,n)=1}}^n a = n^{\phi(n)} \prod_{d|n} \left(\frac{d!}{d^d} \right)^{\mu(n/d)}.$$

8. Let f, g be arithmetic functions.

(a) Show that $Z(f * g, s) = Z(f, s)Z(g, s)$.

(b) Show that there is an arithmetic function f^{-1} such that $f * f^{-1} = f^{-1} * f = \mathbf{1}$ iff $f(1) \neq 0$.

9. (Bonus)

(a) Show that $\sum_{j=1}^n j^2 = n(n+1)(2n+1)/6$.

(b) Let $S(n)$ be the sum of the squares of the positive integers $\leq n$ and coprime to n . Show that

$$\sum_{j=1}^n j^2 = \sum_{d|n} d^2 S(n/d) = \sum_{d|n} \frac{n^2}{d^2} S(d)$$

(c) Combine parts (a) and (b) to show that

$$\sum_{d|n} \frac{S(d)}{d^2} = \frac{1}{6} \left(2n + 3 + \frac{1}{n} \right).$$

Then apply Möbius inversion to show

$$\frac{S(n)}{n^2} = \sum_{d|n} \frac{1}{6} \mu(d) \left(\frac{2n}{d} + 3 + \frac{d}{n} \right).$$

(d) Let $s(n)$ be the the largest square-free divisor of n , i.e. $s(n) = \prod_{p|n} p$. Show that

$$\sum_{d|n} d \mu(d) = (-1)^{\omega(n)} \phi(n) s(n) / n.$$

(e) Finally, conclude that $S(n) = n^2 \phi(n) / 3 + (-1)^{\omega(n)} \phi(n) s(n) / 6$ for $n > 1$.

10. (Bonus)

(a) Show that $\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)}$.

(b) Let $F(n) = \sum_{a=1}^n e^{2\pi i a/n}$. Show that $F(1) = 1$, and $F(n) = 0$ for all $n > 1$.

(c) Show that for all positive integers n ,

$$\sum_{\substack{a=1 \\ (a,n)=1}}^n e^{2\pi i a/n} = \mu(n).$$

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