18.781 Solutions to Problem Set 9

1. We have

$$\begin{aligned} x - \frac{p_{n-1}}{q_{n-1}} &= \frac{x_n p_{n-1} + p_{n-2}}{x_n q_{n-1} + q_{n-2}} - \frac{p_{n-1}}{q_{n-1}} \\ &= \frac{p_{n-2}q_{n-1} - p_{n-1}q_{n-2}}{q_{n-1}(x_n q_{n-1} + q_{n-2})} \\ &= \frac{(-1)^{x-1}}{q_{n-1}(x_n q_{n-1} + q_{n-2})}. \end{aligned}$$

Now as $n \to \infty$ we've shown that $q_n \to \infty$. So

$$\left| x - \frac{p_{n-1}}{q_{n-1}} \right| \to 0$$

from above.

2. Using $n+1$ instead of n in Problem 1, we have

$$\begin{aligned} |q_n x - p_n| &= q_n \left| x - \frac{p_n}{q_n} \right| \\ &= \frac{1}{|x_{n+1} q_n + q_{n-1}|} \\ &= \frac{1}{x_{n+1} q_n + q_{n-1}}, \end{aligned}$$

since $x_{n+1} \geq 1$. We need to show that the RHS is greater than $1/q_{n+2}$. Now $x_{n+1} < a_{n+1} + 1$, so

$$\begin{aligned} x_{n+1} q_n + q_{n-1} &< (a_{n+1} + 1) q_n + q_{n-1} \\ &= a_{n+1} q_n + q_{n-1} + q_n \\ &= q_{n+1} + q_n \\ &\leq a_{n+2} q_{n+1} + q_n \\ &= q_{n+2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| x - \frac{p_{n+1}}{q_{n+1}} \right| &\leq \frac{1}{q_{n+1} q_{n+2}} \\ &< \frac{|x q_n - p_n|}{q_{n+1}} \\ &= \frac{q_n}{q_{n+1}} \left| x - \frac{p_n}{q_n} \right|, \end{aligned}$$

and since $q_{n+1} = a_n q_n + q_{n-1} > q_n$, we get

$$\left| x - \frac{p_{n+1}}{q_{n+1}} \right| < \left| x - \frac{p_n}{q_n} \right|.$$

3. (a) Proceed by contradiction, assuming that $b < q_{n+1}$ and $|bx - a| < |q_n x - p_n|$. As in the hint, we write the vector (a, b) as an integer linear combination of (p_n, q_n) and (p_{n+1}, q_{n+1}). This is possible because the matrix with rows (p_n, q_n) and (p_{n+1}, q_{n+1}) has determinant $(-1)^{n+1}$ and is therefore invertible with the inverse having integer entries. So there are integers y, z such that

$$a = yp_n + zp_{n+1}$$

$$b = yq_n + zq_{n+1}$$

First let's make sure that y and z are nonzero. If $y = 0$ then $b = zq_{n+1}$, which is impossible since $0 < b < q_{n+1}$. If $z = 0$ then

$$|bx - a| = |y||xq_n - p_n| \geq |xq_n - p_n|,$$

contradicting the assumption that $|xb - a| < |xq_n - p_n|$. So both y and z are nonzero.

Next, we'll show that they have opposite signs. If $z > 0$ then

$$yq_n = b - zq_{n+1} \leq b - q_{n+1} < 0,$$

so $y < 0$, and if $z < 0$ then

$$yq_n = b - zq_{n+1} > 0,$$

so $y > 0$. Finally,

$$\begin{aligned} xb - a &= x(yq_n + zq_{n+1}) - (yp_n + zp_{n+1}) \\ &= y(xq_n - p_n) + z(xq_{n+1} - p_{n+1}) \end{aligned}$$

Now we showed that $x - p_n/q_n$ and $x - p_{n+1}/q_{n+1}$ have opposite signs. Since y and z have opposite signs, $y(xq_n - p_n)$ and $z(xq_{n+1} - p_{n+1})$ have the same sign. So

$$\begin{aligned} |bx - a| &= |y(xq_n - p_n) + z(xq_{n+1} - p_{n+1})| \\ &= |y||xq_n - p_n| + |z||xq_{n+1} - p_{n+1}| \\ &\geq |q_n x - p_n|, \end{aligned}$$

contradiction.

(b) Suppose $1 \leq b \leq q_n$. Then $b < q_{n+1}$, so by part (a), $|bx - a| \geq |q_n x - p_n|$. Since $1/b \geq 1/q_n$,

$$\left|x - \frac{a}{b}\right| \geq \left|x - \frac{p_n}{q_n}\right|.$$

4. Suppose a/b is not a convergent. As in the hint, choose n such that $q_n \leq b < q_{n+1}$. (This is possible since the q_i are increasing and go to infinity.) Then

$$\begin{aligned} \left|x - \frac{p_n}{q_n}\right| &= \frac{1}{q_n}|p_n x - q_n| \\ &\leq \frac{1}{q_n}|bx - a| \\ &< \frac{1}{q_n} \cdot \frac{b}{2b^2} \\ &= \frac{1}{2bq_n}. \end{aligned}$$

Now $|aq_n - bp_n| \geq 1$ because by assumption $\frac{a}{b} \neq \frac{p_n}{q_n}$. Hence,

$$\begin{aligned}\frac{1}{bq_n} &\leq \frac{|aq_n - bp_n|}{bq_n} \\ &= \left|\frac{a}{b} - \frac{p_n}{q_n}\right| \\ &\leq \left|\frac{a}{b} - x\right| + \left|x - \frac{p_n}{q_n}\right| \\ &< \frac{1}{2b^2} + \frac{1}{2bq_n}.\end{aligned}$$

This implies that $\frac{1}{2bq_n} < \frac{1}{2b^2}$, so $b < q_n$, contradiction.

5. Problem 4 shows that if p/q satisfies

$$\left|\phi - \frac{p}{q}\right| < \frac{1}{\kappa q^2}, \tag{1}$$

then p/q is a convergent to ϕ, since $\kappa > \sqrt{5} > 2$. So it's enough to show that only finitely many convergents p_n/q_n to ϕ can satisfy this bound.

We showed that the $(n-1)$st convergent to ϕ is just F_{n+1}/F_n. So suppose

$$\left|\phi - \frac{F_{n+1}}{F_n}\right| < \frac{1}{\kappa F_n^2}. \tag{$*$}$$

Now, we claim that

$$\lim_{n\to\infty}\left|\left(\phi - \frac{F_{n+1}}{F_n}\right)F_n^2\right| = \frac{1}{\sqrt{5}}.$$

From this statement it would then follow that for some sufficiently large N,

$$\left|\left(\phi - \frac{F_{n+1}}{F_n}\right)F_n^2\right| > \frac{1}{\kappa}$$

for all $n > N$. Then the only solutions to $(*)$ occur when $n \leq N$, and thus there are finitely many such convergents.

Now let α and β be the roots of $x^2 - x - 1 = 0$, with $\phi = \alpha > \beta$. Since $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$,

$$\begin{aligned}\left(\phi - \frac{F_{n+1}}{F_n}\right)F_n^2 &= \left(\alpha - \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n}\right)\frac{(\alpha^n - \beta^n)^2}{(\alpha - \beta)^2} \\ &= \left(\frac{\beta^n(\beta - \alpha)}{\alpha^n - \beta^n}\right)\frac{(\alpha^n - \beta^n)^2}{(\alpha - \beta)^2} \\ &= \frac{-\beta^n(\alpha^n - \beta^n)}{\alpha - \beta} \\ &= -\frac{(\alpha\beta)^n - \beta^{2n}}{\alpha - \beta} \\ &= -\frac{(-1)^n - \beta^{2n}}{\alpha - \beta}.\end{aligned}$$

We know that $\beta^{2n} \to 0$ as $n \to \infty$ because $|\beta| < 1$. It follows that the magnitude of the RHS approaches $1/\sqrt{5}$, and we are done.

6. (a) Since i is the largest integer such that $q_i \leq \sqrt{p}$, we have $\sqrt{p} < q_{i+1}$. So

$$\left|\frac{p_i}{q_i} - \frac{u}{p}\right| < \frac{1}{q_i q_{i+1}} < \frac{1}{q_i\sqrt{p}}.$$

Multiplying by pq_i, we get the desired bound $|p_i p - uq_i| < \sqrt{p}$.

(b) We have $x = q_i \le \sqrt{p}$, and by part (a), $|y| = |p_i p - u q_i| < \sqrt{p}$, so $x^2 + y^2 < p + p = 2p$. Moreover,

$$\begin{aligned} x^2 + y^2 &\equiv q_i^2 + u^2 q_i^2 \\ &\equiv (u^2+1) q_i^2 \\ &\equiv 0 \pmod{p}. \end{aligned}$$

Clearly, $x^2 + y^2 > 0$, since $x = q_i > 0$. The only multiple of p in $(0, 2p)$ is p, so we must have $x^2 + y^2 = p$.

7. We need to find all c such that $x = (\sqrt{d} + \lfloor \sqrt{d} \rfloor)/c > 1$ and its conjugate $x' = (-\sqrt{d} + \lfloor \sqrt{d} \rfloor)/c$ lies between 0 and -1. The second condition is automatic since the numerator is always between 0 and -1, and c is a positive integer. The first condition holds for all positive integers $c \le 2\lfloor \sqrt{d} \rfloor$.

8. (a) Consider the fractional part $\{ix\}$ of ix as i ranges from 0 through N. Since x is irrational, each $\{ix\}$ is a distinct number in the range $[0, 1)$. In fact, we'll want to wrap the interval into a circle. Consider the $N+1$ segments that the circle is broken up into by the $\{ix\}$. Since the total arclength of the segments is 1, some segment has length no more than $\frac{1}{N+1}$. What this means is that there are two integers $i, j \in \{0, 1, \ldots, N\}$ such that for some integer a

$$0 < |jx - ix + a| \le \frac{1}{N+1}.$$

Setting $q = |i - j| < N$ and $p = a$, division by $i - j$ yields

$$0 < \left| x - \frac{p}{q} \right| \le \frac{1}{q(N+1)},$$

as desired.

(b) First we pick any N_1, and find $q_1 \le N_1$ such that

$$\left| x - \frac{p_1}{q_1} \right| \le \frac{1}{q_1(N_1+1)} < \frac{1}{q_1^2}.$$

Then, since x is irrational, we can pick an N_2 such that

$$\frac{1}{N_2} < \left| x - \frac{p_1}{q_1} \right|.$$

Again using part (a), there exists p_2/q_2 with $q_2 \le N_2$ such that

$$\left| x - \frac{p_2}{q_2} \right| \le \frac{1}{q_2(N_2+1)} < \frac{1}{N_2} < \left| x - \frac{p_1}{q_1} \right|,$$

so p_1/q_1 is distinct from p_2/q_2, and as before

$$\left| x - \frac{p_2}{q_2} \right| \le \frac{1}{q_2(N_2+1)} < \frac{1}{q_2^2}.$$

Picking N_3 with $1/N_3 < |x - p_2/q_2|$ and continuing this process, we can form an infinite series of distinct p_i/q_i such that $|x - p_i/q_i| < 1/q_i^2$.

9. (a) We have that $x = m + \frac{1}{x}$. Solving the quadratic equation and taking the positive root, we get

$$x = \frac{m + \sqrt{m^2+4}}{2}.$$

(b) We know that $p_n = mp_{n-1} + p_{n-2}$, so the characteristic polynomial is $x^2 - mx - 1 = 0$. Thus, letting

$$\alpha = \frac{m + \sqrt{m^2+4}}{2}, \beta = \frac{m - \sqrt{m^2+4}}{2}$$

be the roots of the characteristic polynomial, $p_n = A\alpha^n + B\beta^n$. Using the initial conditions $p_0/q_0 = m/1$ and $p_1/q_1 = (m^2+1)/m$, we can solve the linear system of equations to get

$$\begin{cases} A = \dfrac{\alpha^2}{\sqrt{m^2+4}} \\ B = \dfrac{-\beta^2}{\sqrt{m^2+4}}. \end{cases}$$

So

$$\begin{aligned} p_n &= \frac{1}{\sqrt{m^2+4}}(\alpha^{n+2} - \beta^{n+2}) \\ &= \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}. \end{aligned}$$

Similarly, it can be shown that $q_n = (\alpha^{n+1} - \beta^{n+1})/(\alpha - \beta)$. Therefore,

$$\frac{p_n}{q_n} = \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha^{n+1} - \beta^{n+1}}.$$

10. (a) Suppose we have proven the inequality for $n = 2^{k-1}$. Then

$$\begin{aligned} \frac{r_1 + \cdots + r_{2^k}}{2^k} &= \frac{\left(\frac{r_1 + \cdots + r_{2^{k-1}}}{2^{k-1}}\right) + \left(\frac{r_{2^{k-1}+1} + \cdots + r_{2^k}}{2^{k-1}}\right)}{2} \\ &\geq \frac{(r_1 \cdots r_{2^{k-1}})^{\frac{1}{2^{k-1}}} + (r_{2^{k-1}+1} \cdots r_{2^k})^{\frac{1}{2^{k-1}}}}{2} \\ &\geq \sqrt{(r_1 \cdots r_{2^{k-1}})^{\frac{1}{2^{k-1}}} \cdot (r_{2^{k-1}+1} \cdots r_{2^k})^{\frac{1}{2^{k-1}}}} \\ &= \sqrt[2^k]{r_1 \cdots r_{2^k}}, \end{aligned}$$

completing the induction.

(b) Let $2^{k-1} < n \leq 2^k$, and append $2^k - n$ copies of $r = \frac{r_1 + \cdots + r_n}{n}$. Then the arithmetic mean of $r_1, \ldots, r_n, r, \ldots, r$ is

$$\begin{aligned} \frac{r_1 + \cdots + r_n + (2^k - n)r}{2^k} &= \frac{nr + (2^k - n)r}{2^k} \\ &= \frac{2^k r}{2^k} \\ &= r. \end{aligned}$$

Now part (a) tells us that

$$r \geq (r_1 \cdots r_n \cdot r \cdots r)^{1/2^k},$$

so

$$r^{2k} \geq (r_1 \cdots r_n) \cdot r^{2^k - n},$$

from which it follows that

$$r \geq \sqrt[n]{r_1 \cdots r_n}.$$

Equality holds if and only if $r_1 = r_2 = \cdots = r_n$.

11. We will first define a particular number x (called Liouville's number) which will work for any c. Choose exponents $e_n = n!$ and let $q_n = 10^{e_n}$. Note that for all k, $e_k < e_{k+1}$, so $q_k \mid q_{k+1}$. Now define

$$x = 1 + \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} + \cdots,$$

which converges because $q_n \geq 10^n$ and the geometric series $1 + 1/10 + 1/100 + \cdots$ converges, and let

$$\frac{p_n}{q_n} = 1 + \frac{1}{q_1} + \cdots + \frac{1}{q_n}.$$

The denominator is exactly q_n because each of $q_1, \ldots, q_{n-1}$ must divide q_n. Now

$$\begin{aligned} \left| x - \frac{p_n}{q_n} \right| &= \left| \frac{1}{q_{n+1}} + \frac{1}{q_{n+2}} + \cdots \right| \\ &= \frac{1}{q_{n+1}} \left| 1 + \frac{q_{n+1}}{q_{n+2}} + \frac{q_{n+1}}{q_{n+3}} + \cdots \right| \\ &< \frac{1}{q_{n+1}} \left| 1 + \frac{1}{2} + \frac{1}{4} + \cdots \right| \\ &= \frac{2}{q_{n+1}}. \end{aligned}$$

So we'll be done if we show that for all large enough n,

$$\frac{2}{q_{n+1}} < \frac{1}{q_n^c}.$$

Taking logs base 10, this is equivalent to saying that

$$\log_{10} 2 + c(n!) < (n+1)!,$$

which is obviously true as soon as $n > c + 1$, for instance. Thus, for any c, there are infinitely many rational numbers p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^c}.$$

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