18.781 Practice Questions for Midterm 2

Note: The actual exam will be shorter (about 10 of these questions), in case you are timing yourself.

1. Find a primitive root modulo $343 = 7^3$.

2. How many solutions are there to $x^{12} \equiv 7 \pmod{19}$? To $x^{12} \equiv 6 \pmod{19}$?

3. Solve the congruence $3x^2 + 4x - 2 \equiv 0 \pmod{31}$.

4. Characterize all primes p such that 15 is a square modulo p.

5. If n is odd, evaluate the Jacobi symbol $\left(\frac{n^3}{n-2}\right)$.

6. If $n = p_1^{e_1} \dots p_r^{e_r}$, how many squares modulo n are there? How many quadratic residues modulo n are there (i.e. the squares which are coprime to n)?

7. Let $p > 3$ be a prime. Show that the number of solutions (x, y) of the congruence $x^2 + y^2 \equiv 3 \pmod{p}$ is $p - \left(\frac{-1}{p}\right)$.

8. Compute (with justification) the cyclotomic polynomial $\Phi_{12}(x)$.

9. Let $f(n) = (-1)^n$. Compute
$$Z(f, 2) = \sum_{n \geq 1} \frac{f(n)}{n^2}.$$
(you may use that $\sum 1/n^2 = \pi^2/6$.)

10. For $n = p_1^{e_1} \dots p_r^{e_r}$, calculate the value of $(U * U * U)(n)$, where U is the arithmetic function such that $U(n) = 1$ for all n.

11. Let p be a prime which is 1 mod 4, and suppose $p = a^2 + b^2$ with a odd and positive. Show that $\left(\frac{a}{p}\right) = 1$.

12. Let a_1, a_2, a_3, a_4 be integers. Show that the product $p = \prod_{i<j}(a_i - a_j)$ is divisible by 12.

13. Let the sequence $\{a_n\}$ be given by $a_0 = 0, a_1 = 1$ and for $n \geq 2$,
$$a_n = 5a_{n-1} - 6a_{n-2}.$$
Show that for every prime $p > 3$, we have $p \mid a_p$.

14. Find a positive integer such that $\mu(n) + \mu(n+1) + \mu(n+2) = 3$.

15. Compute the set of integers n for which $\sum_{d|n} \mu(d)\phi(d) = 0$.

16. Let f be a multiplicative function which is not identically zero. Show that $f(1) = 1$.

18.781 Theory of Numbers
Spring 2012