Lecture 18
Continued Fractions I

Continued Fractions - different way to represent real numbers.

$$\frac{415}{93} = 4 + \frac{43}{93} = 4 + \frac{1}{\frac{93}{43}} = 4 + \frac{1}{2 + \frac{7}{43}} = 4 + \frac{1}{2 + \frac{1}{\frac{43}{7}}} = 4 + \frac{1}{2 + \frac{1}{6 + \frac{1}{7}}}$$
$$= [4, 2, 6, 7]$$

In general:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots \frac{1}{a_n}}}} = [a_0, a_1, a_2, \dots a_n]$$

Simple continued fraction if $a_i \in \mathbb{Z}$ and $a_i > 0$ for $i > 0$. Contains the same information as an application of Euclid's Algorithm

$$\begin{aligned} 415 &= 4 \cdot 93 + 43 & &\Rightarrow \frac{415}{93} = 4 + \frac{43}{93} \\ 93 &= 2 \cdot 43 + 7 & &\Rightarrow \frac{93}{43} = 2 + \frac{7}{43} \\ 43 &= 6 \cdot 7 + 1 & &\Rightarrow \frac{43}{7} = 6 + \frac{1}{7} \\ 7 &= 7 \cdot 1 \end{aligned}$$

With this we see that the simple continued fraction of a rational number is always finite. Never terminates for an irrational number.

Eg.

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, \dots]$$

Eg.

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \dots]$$

Eg. Golden Ratio $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$ satisfies $\phi^2 = \phi + 1 \Rightarrow \frac{1}{\phi-1} = \phi$.

$$\phi = 1 + (\phi - 1) = 1 + \cfrac{1}{\cfrac{1}{\phi - 1}} = \frac{1}{1+\phi} = [1, 1, 1, 1, 1, 1, \ldots]$$

Finite simple continued fraction $\iff$ rational number.

Periodic simple continued fraction $\iff$ quadratic irrational (like ϕ)

Eg. What about $\sqrt{2}$? Look at

$$1 + \sqrt{2} = 2 + (\sqrt{2} - 1) = 2 + \cfrac{1}{\cfrac{1}{\sqrt{2} - 1}} = 2 + \cfrac{1}{\cfrac{1+\sqrt{2}}{(1+\sqrt{2})(1-\sqrt{2})}} = 2 + \frac{1}{1+\sqrt{2}}$$

$1 + \sqrt{2} = [2, 2, 2, 2, 2, 2, \ldots]$, so $\sqrt{2} = [1, 2, 2, 2, 2, 2, \ldots]$

What about other algebraic numbers such as $\sqrt[3]{2}$? It's a complete mystery.

(Definition) Convergent: $[a_0, a_1, \ldots a_k]$ is called a **convergent** to $[a_0, a_1, \ldots a_n]$ for $0 \le k \le n$. An infinite simple continued fraction $[a_0, a_1, \ldots]$ equals $\lim_{k\to\infty}[a_0, a_1, \ldots a_k]$. (We will prove this limit exists.)

For $\frac{415}{93}$:

		4	2	6	7
0	1	4	9	58	415
1	0	1	2	13	93

Determinants are $-1, 1, -1, 1, -1$

Recurrence:

$$p_k = a_k p_{k-1} + p_{k-2}$$
$$q_k = a_k q_{k-1} + q_{k-2}$$

		a_0	a_1	a_2	$\ldots$
$p_{-2} = 0$	$p_{-1} = 1$	p_0	p_1	p_2	$\ldots$
$q_{-2} = 1$	$q_{-1} = 0$	q_0	q_1	q_2	$\ldots$

Theorem 59.

$$[a_0, a_1 \ldots a_k] = \frac{p_k}{q_k}$$

Proof. By induction, base case $k = 0$

$$\frac{p_0}{q_0} = \frac{a_0 p_{-1} + p_{-2}}{a_0 q_{-1} + q_{-2}} = \frac{a_0 \cdot 1}{1} = a_0$$

Now assume holds for all k:

$$\begin{aligned}
[a_0, a_1, \dots a_{k+1}] &= \left[a_0, a_1, \dots a_{k-1}, a_k + \frac{1}{a_{k+1}}\right] \\
&= \frac{p'_k}{q'_k} \\
&= \frac{\left(a_k + \frac{1}{a_{k+1}}\right) p'_{k-1} + p'_{k-2}}{\left(a_k + \frac{1}{a_{k+1}}\right) q'_{k-1} + q'_{k-2}} \\
&= \frac{(a_k a_{k+1} + 1) p'_{k-1} + a_{k+1} p'_{k-2}}{(a_k a_{k+1} + 1) q'_{k-1} + a_{k+1} q'_{k-2}} \\
&= \frac{a_{k+1}(a_k p'_{k-1} + p'_{k-2}) + p'_{k-1}}{a_{k+1}(a_k q'_{k-1} + q'_{k-2}) + q'_{k-1}} \\
&= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} \\
&= \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k-1}} \\
&= \frac{p_{k+1}}{q_{k+1}}
\end{aligned}$$

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Theorem 60.

$$p_{k-1} q_k - q_{k-1} p_k = (-1)^k$$

Proof. By induction, base case is easy to check. Assume to hold for k

$$\begin{aligned}
p_k q_{k+1} - q_k p_{k+1} &= p_k(a_{k+1} q_k + q_{k-1}) - q_k(a_{k+1} p_k + p_{k-1}) \\
&= p_k q_{k-1} - q_k p_{k-1} \\
&= -(q_k p_{k-1} - p_k q_{k-1}) \\
&= (-1)(-1)^k \\
&= (-1)^{k+1}
\end{aligned}$$

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Proof 2.

$$\begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix} \begin{bmatrix} a_{k+1} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{k+1}p_k + p_{k-1} & p_k \\ a_{k+1}q_k + q_{k-1} & q_k \end{bmatrix} = \begin{bmatrix} p_{k+1} & p_k \\ q_{k+1} & q_k \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} &= \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix} \cdots \\ &= \prod_{k=0}^{n} \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix} \\ \begin{vmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{vmatrix} &= \prod_{k=0}^{n} \begin{vmatrix} a_k & 1 \\ 1 & 0 \end{vmatrix} \\ &= (-1)^{n+1} \end{aligned}$$

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Note: Take the transpose

$$\begin{bmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} = \prod_{k=n}^{0} \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix}$$

We get that

$$\begin{aligned} \frac{p_n}{p_{n-1}} &= [a_n, a_{n-1}, \ldots a_0] \\ \frac{q_n}{q_{n-1}} &= [a_n, a_{n-1}, \ldots a_1] \end{aligned}$$

Corollary 61.

$$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}}$$

Corollary 62.

$$(p_k, q_k) = 1$$

Corollary 63.

$$p_{k-2}q_k - q_{k-2}p_k = (-1)^{k-1}a_k$$

Proof.

$$\begin{aligned} p_{k-1}q_k - q_{k-1}p_k &= (-1)^k \\ a_k p_{k-1}q_k - a_k q_{k-1}p_k &= (-1)^k a_k \\ (p_k - p_{k-2})q_k - (q_k - q_{k-2})p_k &= (-1)^k a_k \\ p_{k-2}q_k - q_{k-2}p_k = (-1)^{k+1} a_k \end{aligned}$$

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$$\text{Corollary 3} \Rightarrow \frac{p_{k-2}}{q_{k-2}} - \frac{p_k}{q_k} = \frac{(-1)^{k-1}a_k}{q_{k-2}q_k} = \begin{cases} < 0 & k \text{ even} \\ > 0 & k \text{ odd} \end{cases}$$
$$\Rightarrow \frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} \dots$$
$$\Rightarrow \frac{p_1}{q_1} > \frac{p_3}{q_3} > \frac{p_5}{q_5} \dots$$

Even terms increasing, bounded above by odd terms, odd terms decreasing, bounded below by even terms, so they both converge. From Corollary 1 the even and odd convergents get arbitrarily close. So both even and odd sequences converge to the same real number x.

$$\left| \frac{p_k}{q_k} - x \right| \leq \left| \frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} \right| = \frac{1}{q_k q_{k-1}} \leq \frac{1}{q_k^2}$$

$\Rightarrow$ very good approximations.

Theorem 64. *One of every* 2 *consecutive convergents satisfies*

$$\left| \frac{p_k}{q_k} - x \right| \leq \frac{1}{2q_k^2}$$

Theorem 65. *One of every* 3 *consecutive convergents satisfies*

$$\left| \frac{p_k}{q_k} - x \right| \leq \frac{1}{\sqrt{5}q_k^2}$$

(Proofs in next lecture)