Lecture 19

Continued Fractions II: Inequalities

Real number x, compute integers $a_0, a_1, \ldots$ such that $a_0 = \lfloor x \rfloor$,

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 \ddots}}$$

Let $x_1 = \frac{1}{x - a_0}$, real number ≥ 1 as long as well defined, $a_1 = \lfloor x_1 \rfloor, x_2 = \frac{1}{x_1 - a_1}$.

For $i > 0, a_i \geq 1$.

Convergents $\frac{p_n}{q_n} = [a_0, a_1, \ldots a_n]$. $\frac{p_n}{q_n} \to x$ as $n \to \infty$.

$$\frac{p_0}{q_0} < \frac{p_2}{q_4} < \cdots < x < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}$$

$$\left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| = \frac{1}{q_n q_{n+1}} \to 0 \text{ as } n \to \infty$$

so

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} \leq \frac{1}{q_n^2}$$

Why are continued fractions useful/interesting?

1. Gives good approximations to real numbers
2. Continued fractions and higher dimensional variants have applications in engineering
3. Useful in number theory for study of quadratic fields, diophantine equations

Theorem 66. *One of every two consecutive convergents satisfies*

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{2q_n^2}$$

Proof.

$$\left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| = \frac{1}{q_n q_{n+1}} \leq \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2}$$

using AM-GM inequality with $\frac{1}{q_n^2}$ and $\frac{1}{q_{n+1}^2}$

$$\left| x - \frac{p_n}{q_n} \right| + \left| x - \frac{p_{n+1}}{q_{n+1}} \right| = \left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| \leq \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2}$$

$$\Rightarrow \left|x - \frac{p_n}{q_n}\right| \leq \frac{1}{2q_n^2} \text{ or } \left|x - \frac{p_{n+1}}{q_{n+1}}\right| \leq \frac{1}{2q_{n+1}^2}$$

■

Theorem 67. *One of every three consecutive convergents satisfies*

$$\left|x - \frac{p_n}{q_n}\right| \leq \frac{1}{\sqrt{5}q_n^2}$$

Proof. Suppose not, and that

$$\left|x - \frac{p_n}{q_n}\right| \leq \frac{1}{\sqrt{5}q_n^2} \text{ for } n, n+1, n+2$$

$$\begin{aligned}
\left|x - \frac{p_n}{q_n}\right| + \left|x - \frac{p_{n+1}}{q_{n+1}}\right| &= \left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| \\
&= \frac{1}{q_n q_{n+1}} \\
&> \frac{1}{\sqrt{5}q_n^2} + \frac{1}{\sqrt{5}q_{n+1}^2} \\
\Rightarrow \sqrt{5} &> \frac{q_{n+1}}{q_n} + \frac{q_n}{q_{n+1}} \\
\Rightarrow \frac{q_{n+1}}{q_n} &< \frac{\sqrt{5}+1}{2}
\end{aligned}$$

using the fact that $f(x) = x + \frac{1}{x}$ is strictly increasing on $(1, \infty)$

$$\Rightarrow \frac{q_n}{q_{n+1}} = \frac{1}{\frac{q_n}{q_{n+1}}} > \frac{1}{\frac{\sqrt{5}+1}{2}} = \frac{\sqrt{5}-1}{2}$$

Same argument for $n+1$ and $n+2$ says that $\frac{q_{n+2}}{q_{n+1}} < \frac{\sqrt{5}+1}{2}$.

$$\begin{aligned}
\frac{q_{n+2}}{q_{n+1}} &= \frac{a_{n+2}q_{n+1} + q_n}{q_{n+1}} \\
&= a_{n+2} + \frac{q_n}{q_{n+1}} \\
&\geq 1 + \frac{\sqrt{5}-1}{2} \\
&= \frac{\sqrt{5}+1}{2}
\end{aligned}$$

leading to a contradiction ($\lightning$) ■

Corollary 68. *For any irrational real number x, there are infinitely many rational numbers $\frac{p}{q}$ such that $\left|x - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$*

Proof. Write convergents as

$$\underbrace{\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}}_{\text{one satisfies}}, \underbrace{\frac{p_4}{q_4}, \frac{p_5}{q_5}, \frac{p_6}{q_6}}_{\text{one satisfies}},, \dots$$

■

Theorem 69. *$\sqrt{5}$ is optimal (cannot be replaced by any larger value) - ie., there does not exist an $\alpha < \sqrt{5}$ such that for any irrational x there are infinitely many rational numbers $\left|x - \frac{p}{q}\right| < \frac{1}{\alpha q^2}$*

Proof. We won't prove this here [proved in PSet 9], but we'll give a heuristic argument for why $\sqrt{5}$ is the best.

Consider $\alpha = \frac{1+\sqrt{5}}{2} =$ golden ratio. It has the continued fraction $[1, 1, 1, 1, \dots]$, and convergents are $1, 1 + \frac{1}{1} = 2, 1 + \frac{1}{2} = \frac{3}{2}, 1 + \frac{2}{3} = \frac{5}{3}, 1 + \frac{3}{5} = \frac{8}{5} \dots$. By induction they are ratios of consecutive Fibonacci numbers.

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}$$

We'll show

$$\left|\frac{F_{n+1}}{F_n} - \alpha\right| \cdot F_n^2 \to \frac{1}{\alpha - \beta} = \frac{1}{\sqrt{5}} \text{ as } n \to \infty$$

$$\begin{aligned}
\left|\frac{F_{n+1}}{F_n} - \alpha\right| \cdot F_n^2 &= \left|\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} - \alpha\right| \cdot \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^2 \\
&= \left|\frac{\alpha^{n+1} - \beta^{n+1} - \alpha^{n+1} + \beta^n\alpha}{\alpha^n - \beta^n}\right| \frac{|\alpha^n - \beta^n|^2}{|\alpha - \beta|^2} \\
&= \frac{\beta^n|\alpha - \beta||\alpha^n - \beta^n|}{|\alpha - \beta|^2} \\
&= \frac{|(\beta\alpha)^n - \beta^{2n}|}{|\alpha - \beta|} \\
&= \frac{|(-1)^n - \beta^{2n}|}{|\alpha - \beta|}
\end{aligned}$$

Since $|\beta| < 1, \beta^{2n} \to 0$ as $n \to \infty$, so expression tends to $\frac{1}{\alpha - \beta} = \frac{1}{\sqrt{5}}$ as $n \to \infty$. ■

Theorem 70. *A real number x is a quadratic irrational (ie., $x = r + s\sqrt{t}$ where $r, s \in \mathbb{Q}$, and t is a squarefree integer) if and only if its continued fraction is periodic ($x = [b_0, b_1, \ldots b_k, a_0, a_1, \ldots a_{n-1}, a_0, a_1, \ldots a_{n-1}, \ldots] = [b_0, \ldots b_k, \overline{a_0, \ldots a_{n-1}}]$.)*

Proof - Part 1. Suppose $x = [b_0, b_1, \ldots b_k, \overline{a_0, a_1, \ldots a_{n-1}}]$, let $\theta = [\overline{a_0, \ldots a_{n-1}}]$.

$$\theta = [a_0, a_1, \ldots a_{n-1}, \theta] = \frac{p_{n-1}\theta + p_{n-2}}{q_{n-1}\theta + q_{n-2}}$$

for some positive $p_{n-1}, p_{n-2}, q_{n-1}, q_{n-2}$, which leads to a quadratic equation for θ. θ irrational because it's an infinite continued fraction. Then $x = [b_0, \ldots, b_k, \theta]$ is also a quadratic irrational. □

Proof - Part 2. Want to show that if $x = \frac{a+\sqrt{b}}{c}$, where a, b, c are integers, $b > 0, c \neq 0, b$ not a perfect square, then continued fraction of x is periodic.

Step 0: We can write this as

$$x = \frac{ac + \sqrt{bc^2}}{c^2} \text{ if } c > 0, \text{ or } \frac{-ac + \sqrt{bc^2}}{-c^2} \text{ if } c < 0$$

In either case, $bc^2 - (\pm ac)^2 = c^2(b - a^2)$, which is divisible by $\pm c^2$. In either case, we've written x as

$$x = \frac{B_0 + \sqrt{d}}{C_0}, \text{ with } C|d - B_0^2$$

Fix such an expression (in particular, d).

Step 1: Let $x_0 = x$ and define by induction

$$\begin{aligned} a_i &= \lfloor x_i \rfloor \\ x_i &= \frac{B_i + \sqrt{d}}{C_i} \\ x_{i+1} &= \frac{1}{x_i - a_i} \\ B_{i+1} &= a_i C_i - B_i \\ C_{i+1} &= \frac{d - B_{i+1}^2}{C_i} \end{aligned}$$

So far, we know B_i, C_i are rational numbers. Strategy will be to show that B_i, C_i are integers, and that they're bounded in absolute value - use this to show that they repeat.

Definitions of B_{i+1}, C_{i+1} motivated by

$$
\begin{aligned}
x_i &= \frac{B_i + \sqrt{d}}{C_i} \\
x_{i+1} &= \frac{1}{x_i - a_i} \\
&= \frac{1}{\frac{B_i + \sqrt{d}}{C_i} - a_i} \\
&= \frac{C_i}{\sqrt{d} - (a_i C_i - B_i)} \\
\frac{B_{i+1} + \sqrt{d}}{C_{i+1}} &= \frac{C_i(\sqrt{d} + a_i C_i - B_i)}{d - (a_i C_i - B_i)^2} \\
&= \frac{a_i C_i - B_i + \sqrt{d}}{\frac{d - B_{i+1}^2}{C_i}}
\end{aligned}
$$