

## 18.786 Detachable Midterm: April 6, 2006

Please turn in at least 2 of the following four problems during the class period. Please turn in the remaining problems on Thursday, April 13 (no extensions!). Please do not discuss the midterm problems until after the detached problems come due. Also, please write your name on *each problem page*; you may as well do this now.

This midterm is “open materials”; during the class period, you may use Janusz’s book, your course notes, or your problem sets. For the detached problems, you may use any resource you want except for another human being; this includes computer programs. Keep this in mind as you decide which problems to work on here. (Warning: the fact that “SAGE says that the answer is xxx” does not by itself constitute an adequate justification! All proofs must be readable by a sufficiently diligent human, namely me.) Each problem has 3 parts; each part of each problem is equally weighted.

The rest of this page is intentionally blank; you may use it for scratch work. Please do not turn in this page; put all work to be submitted on the problem sheets (you may use the back of the page).

1. Define the polynomial  $P(x) = x^3 + 2x^2 - 4x + 2$ ; it can be shown that  $P$  is irreducible and its discriminant is  $-140$ .

- (a) Compute the signature (number of real and complex embeddings) of  $P$ .
- (a) Check that the ring of integers of  $K = \mathbb{Q}[x]/(P(x))$  is equal to  $\mathbb{Z}[x]$ . (Hint: shorten the computation using results from class and/or Janusz.)
- (c) Determine the factorization (with values of  $e$  and  $f$ ) in  $\mathfrak{o}_K$  of the ideals  $(2)$ ,  $(3)$ ,  $(5)$ ,  $(7)$ .

2. Write down examples of the following. (You do not need to provide more than one sentence of justification.)

- (a) Two number fields (not necessarily of the same degree) with the same discriminant (including the sign).
- (b) A number field of degree 12 ramified only above 2 and 3.
- (c) Two number fields  $K$  and  $L$  which are linearly disjoint over  $\mathbb{Q}$ , such that the compositum  $LK$  is everywhere unramified over  $K$ .

3. Let  $K$  be the number field  $\mathbb{Q}(\sqrt{2}, \sqrt{-5})$ .

- (a) Prove that the group of units of  $K$  has rank 1.
- (b) Prove that the torsion subgroup of  $\mathfrak{o}_K^*$  is  $\{\pm 1\}$ . (Hint: you can use ramification considerations to restrict the order.)
- (c) Find an element of  $\mathfrak{o}_K^*$  of infinite order. (Hint: look at subfields.)

4. Let  $K$  be the number field  $\mathbb{Q}[x]/(x^3 - 5)$ . It can be shown that  $\mathfrak{o}_K = \mathbb{Z}[x]/(x^3 - 5)$ ; you may assume this without proof.

- (a) Compute the discriminant of  $K$ .
- (b) Compute a constant  $C$  such that for every ideal  $\mathfrak{a}$  of  $\mathfrak{o}_K$ , there is an element  $\alpha$  of  $\mathfrak{a}$  with  $|\text{Norm}(\alpha)| \leq C \text{Norm}(\mathfrak{a})$ .
- (c) Prove that the class group of  $\mathfrak{o}_K$  is trivial. (Hint: the hard part is to check that there are no nonprincipal ideals of norm 2 or 3.)