

18.786 Final exam (take-home, due May 13 at end of lecture)

Please turn in exactly **eight** of the following eleven problems. Each problem is worth 10 points. You may consult any resources (textbook, notes, the internet, software, problem sets), but not other people. In particular, you may not collaborate on this assignment, though you may ask me for clarification or help.

- Let A be a Dedekind domain, and \mathfrak{a} a non-zero ideal of A .
 - Show that A/\mathfrak{a} is a product of principal ideal rings.
 - Show that \mathfrak{a} can be generated as an ideal by two elements.
- Show (without using class field theory) that for any number field K , there exists a finite extension L/K such that every ideal of K becomes a principal ideal of L .
- Let K be a totally real number field of degree n , and let $\sigma_1, \dots, \sigma_n$ be the embeddings of K into \mathbb{R} . Now let k be an integer such that $0 < k < n$. Show that there is a unit ϵ of \mathcal{O}_K satisfying $0 < \sigma_j(\epsilon) < 1$ for $1 \leq j \leq k$ and $\sigma_j(\epsilon) > 1$ for $k+1 \leq j \leq n$.
- Let L/K be a finite extension of number fields. Show that a prime \mathfrak{p} of K is totally split in L if and only if it is totally split in the Galois closure N of L .
 - Let L/K be a Galois extension of number fields with Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Show that there are at most finitely many primes of K which are non-split in L (i.e. only one prime of L lies above it).
- Let A be a finite abelian group. Show that there exists an abelian extension L/\mathbb{Q} with Galois group A .
- Let $p \neq q$ be primes of \mathbb{Q} .
 - Show that the fields \mathbb{Q}_p and \mathbb{Q}_q are non-isomorphic.
 - Show that the only automorphism of \mathbb{Q}_p is the identity.
- Show there is an exact sequence

$$0 \rightarrow \hat{\mathbb{Z}} \rightarrow \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 0$$

of abelian topological groups. Show that $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ is compact, connected and uniquely divisible, i.e. the multiplication by n map is a group isomorphism.

- Let $\mathfrak{m} \mid \mathfrak{n}$ be moduli. Find an exact sequence connecting $C_{\mathfrak{m}}$ with $C_{\mathfrak{n}}$.
- Let L_1, \dots, L_r be cyclic extensions of prime degree p of a number field K , such that $L_i \cap L'_i = K$ for all i , where L'_i is the compositum of the L_j for $j \neq i$. Show that there are infinitely many primes of K which are nonsplit in L_1 but split completely in the other L_i . Is the statement still true if we relax the condition to $L_i \cap L_j = K$ for $i \neq j$?
 - Determine the class group of $K = \mathbb{Q}(\alpha)$ where α satisfies $x^5 - x + 1 = 0$.
 - Let K be a number field and m a natural number. Suppose that for all but finitely many primes \mathfrak{p} of \mathcal{O}_K , the completion $K_{\mathfrak{p}}$ contains a primitive m 'th root of unity. Show that K contains a primitive m 'th root of unity. Suppose $\ell > [K : \mathbb{Q}]$. Show that the number of roots of unity in K is equal to the greatest common divisor of the numbers $N(\mathfrak{p}) - 1$, where \mathfrak{p} ranges over all primes of K for which $\mathcal{O}_K/\mathfrak{p}$ has characteristic greater than ℓ .

11. Let n be a positive integer and let S_n the symmetric group on n letters.

- (a) Let $G \subset S_n$ be a transitive subgroup that is generated by transpositions. Show that $G = S_n$.
- (b) Let K be a finite extension of degree n of \mathbb{Q} . Suppose that the discriminant of K over \mathbb{Q} is squarefree. Show that the Galois closure of L of K has degree $n!$ over \mathbb{Q} , with Galois group S_n .

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