18.786 Problem Set 2 (due Thursday Feb 18)

1. Show that for p a prime, $r \geq 1$, the polynomial $\frac{X^{p^r}-1}{X^{p^{r-1}}-1}$ is irreducible in $\mathbb{Q}[x]$. Compute the ring of integers of $\mathbb{Q}(\zeta_{p^r})$ and therefore the discriminant of this cyclotomic field.

2. Let K be a number field. Show that $\alpha \in \mathcal{O}_K$ is a unit iff $\mathrm{Nm}_{K/\mathbb{Q}}(\alpha) = \pm 1$. Let $p \neq q$ be primes. Show that $\zeta_{pq} - 1 \in \mathcal{O}_{\mathbb{Q}(\zeta_{pq})}$ is a unit.

3. Give an explicit example of a non-perfect field K and a finite extension L of K, with a basis $x_1, \ldots, x_n$ of L over K, such that $D(x_1, \ldots, x_n) = 0$.

4. Compute the units of the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ for $d < 0$.

5. Compute the ring of integers of $\mathbb{Q}(\sqrt[3]{2})$.

6. What is the unique quadratic subfield of $\mathbb{Q}(\zeta_p)$?

7. Let $B \subset B'$ be A-algebras (i.e. there is a ring homomorphism $A \to B$), with B' integral over B, and let C be another A-algebra. Show that $B' \otimes_A C$ is integral over $B \otimes_A C$.

8. Prove the Noether normalization lemma: Let k be a field, and A be a finitely generated algebra over k. Then there exist $x_1, \ldots, x_n \in A$ which are algebraically independent over k and such that A is integral over $k[x_1, \ldots, x_n]$.

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