18.786 Problem Set 5 (due Thursday Mar 11 in class)

1. Compute, with proof, the class group of the ring of integers of $Q(\sqrt{-5})$.

2. We saw in lecture that if α is an algebraic number all of whose conjugates have absolute value 1, then α is a root of unity. Now suppose x is a real algebraic integer such that $\alpha > 1$ and all other conjugates of α lie on or inside the unit circle, and at least one lies on the unit circle. Such an α is called a Salem number. Show that the minimal polynomial $p(x)$ of α is reciprocal, i.e. $x^{\deg p} p(1/x) = p$. Give three examples of Salem numbers.

3. Let $x \in \mathbb{Q}$. Let $\mathcal{M}(\mathbb{Q}) = \{\infty, 2, 3, 5, \dots\}$ be the set of normalized valuations of $\mathbb{Q}$. Show that $\prod_{v \in \mathcal{M}} |x|_v = 1$.

4. Let K be a field which is complete with respect to an archimedean valuation $| \, |$. Show that there is an isomorphism of K with $\mathbb{R}$ or $\mathbb{C}$ which identifies the given valuation on K with a valuation equivalent to the usual valuation on these fields. [Hint: show that you can assume w.lo.g that $\mathbb{R} \subset K$ and then show that every element of K is algebraic over $\mathbb{R}$ of degree at most 2.]

5. Compute a generator for the units of the real quadratic fields $\mathbb{Q}(\sqrt{137})$ and $\mathbb{Q}(\sqrt{139})$.

6. Let $\zeta = e^{2\pi i/11}$ be a primitive eleventh root of unity, and $K = \mathbb{Q}(\zeta)$. Give an explicit basis for a finite index subgroup of the group of units of $\mathcal{O}_K$. Check your answer using the logarithmic embedding (and gp).

18.786 Topics in Algebraic Number Theory
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