18.786 Problem Set 7 (due Thursday Apr 8 in class)

1. Let L, M be finite extensions of a field K.

   (a) If L, M are Galois over K, then so is their compositum LM.

   (b) If L, M are Abelian over K, so is LM.

   (c) If K is a number field, $\mathfrak{p}$ a prime of $\mathcal{O}_K$ which is unramified in L and M, then $\mathfrak{p}$ is unramified in LM.

2. Prove that $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/N\mathbb{Z} \cong \prod_p \mathbb{Z}_p$. (Hint: use unique factorization and the Chinese remainder theorem.)

3. Let L/K be a finite extension of number fields. Let v be an absolute value on K (archimedean or nonarchimedean), and let K_v be the completion of K with respect to v.

   (a) Show that every extension w to L of the valuation v arises as the composite $\bar{v} \circ \tau$ for some K-embedding $\tau : L \to \overline{K_v}$ into the algebraic closure of K_v (here $\bar{v}$ is the unique extension of v to the algebraic closure), and that two such extensins $\bar{v} \circ \tau$ and $\bar{v} \circ \tau'$ are equal iff τ and τ' are conjugate over K_v.

   (b) Show that $L \otimes K_v \cong \prod_{w|v} L_w$, where the product is over all valuations w which extend v. When v is non-archimedean corresponding to the prime $\mathfrak{p}$ of $\mathcal{O}_K$, the w are in one-to-one correspondence with the primes $\mathfrak{P}$ lying above $\mathfrak{p}$. (Hint: Use Proposition 2 of Samuel, section 5.2 to show this.)

   (c) If L/K is Galois then show that all the extensions are conjugates. For $G = Gal(L/K)$, let $G_w = \{g \in G \,|\, gw = w\}$. Show that L_w is Galois over K_v and G_w is its Galois group.

4. Let K be a nonarchimedean local field and $\mathcal{O}_K$ its valuation ring. Let $U = \mathcal{O}_K^\times$ be the units of $\mathcal{O}_K$. Endow $\mathcal{O}_K$ and U with the metric/topology induced from the valuation on K. Show that U is compact, and open and closed in $\mathcal{O}_K$. Show that a subgroup of the additive group $\mathcal{O}_K$ is open iff it is of finite index, and the same statement for the multiplicative group U.

5. Show that cubic field K generated over $\mathbb{Q}$ by a root of $x^3 - x^2 - 2x - 8$ is not monogenic. (Hint: figure out how 2 splits in K, and argue by contradiction.)

6. Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$. Show that $\mathbb{C}_p$ is algebraically closed. (Hint: use Krasner's lemma.)

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