18.786 Problem Set 8 (due Thursday Apr 15 in class)

For problems 4 through 6 you may assume that the abelian category is Mod_R, the category of modules over a ring R.

1. Check the proof of Lemma II.1.2 in Milne's book, with all the details. In particular, check also that induction is indeed a functor.

2. Prove Remark II.1.14 in the book.

3. Prove the normal basis theorem: if L/K is a Galois extension of fields, then $\exists\, y \in L$ such that the set $\{gy \,|\, g \in G = Gal(L/K)\}$ is a basis for L as a K-vector space.

4. Let $\mathcal{C}$ be an Abelian category with enough injectives, and let
$$0 \to X \to Y \to Z \to 0$$
be a short exact sequence in $\mathcal{C}$. Show that there exist injective resolutions $X \to I^\bullet$, $Y \to J^\bullet$ and $Z \to K^\bullet$ fitting into an exact sequence of complexes
$$0 \to I^\bullet \to J^\bullet \to K^\bullet \to 0.$$
(Hint: take any injective resolutions $I^\bullet$ and $K^\bullet$ of X and Z respectively. Then define $J^\bullet = I^\bullet \oplus K^\bullet$ and show how to define maps to make it an injective resolution of Y.)

5. Prove Lemma II.A.10 in the book.

6. Prove Proposition II.A.11 in the book.

7. Let $K = \mathbb{F}_p((T))$, the Laurent series in one variable over $\mathbb{F}_p$. Define the valuation on this field by $|\sum_{r=-m}^{\infty} a_r T^r| = c^m$ for some choice of $c > 1$. That is, the uniformizer is T and the valuation ring consists of the power series in T. Show that not all finite index subgroups of $K^\times$ are open.

8. Let E be the forgetful functor from the category of G-modules to the category of abelian groups, and F the functor from abelian groups to G-modules taking an abelian group A to the G-module $\mathrm{Hom}(\mathbb{Z}[G], A)$. Show that E is left adjoint to F, i.e.
$$\mathrm{Hom}(E(X), Y) \cong \mathrm{Hom}_G(X, F(Y))$$
for a G-module X and an abelian group Y.

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