18.786 Problem Set 9 (due Thursday Apr 22 in class)

1. Let L/K be a finite extension of finite fields. Show that the norm from L to K is surjective. (Hint: use Hilbert's theorem 90 and the Herbrand quotient.)

2. Let L/K be a finite extension of finite fields. Show that the trace map from L to K is surjective.

3. Check that $M_G \cong M \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ where $\mathbb{Z}$ is considered as a trivial $\mathbb{Z}[G]$ module. (Hint: normal basis theorem).

4. Prove Proposition 3.2 in the book.
   Note: for homology, the corestriction map is natural and defined as the map induced by defining $Cor : H_0(H, M) \to H_0(G, M)$ in dimension 0 as $M_H = M/I_H M \to M/I_G M = M_G$, noting that $I_H \subset I_G$, and extending to higher dimensions by using Shapiro's lemma.
   On the other hand, restriction in dimension 0 is $M_G \to M_H$ given by $m \mapsto \sum_{s \in S} s^{-1} m$, where $G = \bigcup_{s \in S} sH$.

   (Hint: Consider the exact sequence of G or H modules $0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ and take homology with respect to G and H and compare).

5. Prove that the Galois group of a finite extension of local fields is solvable, as follows. Let L/K be a finite Galois extension with Galois group G, with v_K a discrete normalized valuation of K which therefore admits a unique extension w to L. Let $v_L = ew$ be the associated normalized valuation of L, where e is the ramification index of L/K (i.e. we want $v_K(\pi_K) = v_L(\pi_L) = 1$).

   For every real number $s \geq -1$ define the $s'th$ ramification group of L/K by

   $$G_s = \{g \in G \,|\, v_L(ga - a) \geq s + 1 \ \forall a \in \mathcal{O}_L\}.$$

   (a) Prove that the G_s form a chain $G = G_{-1} \supset G_0 \supset G_1 \subset \ldots$ of normal subgroups of G.

   (b) Show G_{-1}/G_0 is cyclic.

   (c) For every integer $s \geq 0$, define the map $G_s/G_{s+1} \to U_L^{(s)}/U_L^{(s+1)}$ by sending g to $g(\pi_L)/\pi_L$. (Here $U_L^{(s)} = 1 + m_L^s$ for $s \geq 1$ and $\mathcal{O}_L^*$ for $s = 0$.) Show that this is a well-defined injective homomorphism independent of the choice of uniformizer π_L.

   (d) Show that G_s/G_{s+1} is a finite abelian group for every $s \geq 1$. Conclude that G is solvable.

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