18.786 Problem Set 10 (due Thursday Apr 29 in class)

1. Let K be a nonarchimedean local field.
   (a) Let L be finite and unramified over K. Show that L is Galois over K, and in fact cyclic.
   (b) Show that K^{un} is obtained by adjoining to K all the prime to p roots of unity.

2. Check the details of the comment in Milne pg. 105, that Cor commutes with the isomorphism $H^2_T(\mathrm{Gal}(L/K), \mathbb{Z}) \to H^2_T(\mathrm{Gal}(L/K), L^\times)$, given by cup product with the fundamental class.

3. (Kummer theory) Let K be a nonarchimedean local field, K^{al} its algebraic closure, and $G_K = \mathrm{Gal}(K^{al}/K)$. For any integer n, let μ_n be the group of n'th roots of unity in K^{al}. Assume that $\mu_n \subset K$. Then μ_n is a (trivial) G_K-module. Show that there is an isomorphism of groups $K^\times/(K^\times)^n \to \mathrm{Hom}(G_K, \mu_n)$.

   (Hint: use the short exact sequence

$$0 \to \mu_n \to K^{al} \xrightarrow{(\cdot)^n} K^{al} \to 0$$

   and compute the boundary map in cohomology).

4. Show that for $K = \mathbb{Q}_p, L = K(\zeta_{p^n})$, the image of the norm $N_{L/K}L^\times$ is $U_K^{(n)} \times p^{\mathbb{Z}}$ inside $K^\times = U_K \times p^{\mathbb{Z}}$ (Hint: use Local class field theory to figure out the index of the norm subgroup).

5. (Hilbert symbol) Let K be a nonarchimedean local field. For $a, b \in K^\times$, define $(a, b) = 1$ if $z^2 = ax^2 + by^2$ has nontrivial solution in K, and $(a, b) = -1$ otherwise.
   (a) Show that $(a, b) = 1$ iff b is a norm from $K(\sqrt{a})$.
   Note: If a is a square, define $K(\sqrt{a})$ to be $K[X]/(X^2 - a) \cong K \times K$, with the norm taking (x_1, x_2) to $x_1 x_2$.
   (b) Show that (a, b) is bilinear in each variable: $(aa', b) = (a, b)(a', b)$. (Hint: what is the index of the norm subgroup?).
   (c) Show that $(a, 1 - a) = (a, -a) = 1$.
   (d) Let $p > 2$ be prime. Show that if $a, b \in \mathbb{F}_p^\times$, there are $x, y \in \mathbb{F}_p$ such that $1 = ax^2 + by^2$.
   (e) Compute a table of the Hilbert pairing

$$(\ ,\ ) : \frac{K^\times}{(K^\times)^2} \times \frac{K^\times}{(K^\times)^2} \to \{1, -1\}$$

   for $K = \mathbb{Q}_p, p \neq 2$.

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