42 Tossing a coin (probability)

The result of n tosses of a coin can be represented by a binary number in the interval $[0,1]$ with n digits: the kth digit is 0 if the kth toss comes up tails and 1 if it comes up heads. In binary expansion, $.1 = \frac{1}{2}$, and $.011 = \frac{3}{8}$ stands for the result of three tosses coming up tails, heads, heads. Similarly, an infinite series of tosses gives us a binary expansion of any real number in the interval $[0,1]$. The correspondence between infinite sequences and real numbers is not quite bijective because there are some real numbers with two binary expansions, for instance $.01111\cdots = .10000\cdots$.

Let y be the outcome of an infinite toss, and consider the function on $[0,1]$:

$$P(x) = \text{ probability that } y \leq x.$$

Do not assume that the coin is fair. Instead, let the probability of heads be p, so that the probability of tails is $q = 1 - p$.

Assignment

1. Let x be a point x which has a finite binary expansion. Determine $P(x)$, and explain how it is determined by a finite number of coin tosses.

2. We can approximate P, replacing an infinite sequence of tosses by a sequence of n tosses, interpolated linearly. Thus the first approximation, a single toss, gives us the values of a function at three points: $P(0) = q$, $P(.1) = 1$. The linear interpolation is the function $A_1(x) = q + 2px$ if $x < \frac{1}{2}$ and $A_1(x) = 1$ if $x \geq \frac{1}{2}$. Plot the approximations to the graph of P that are obtained by linear interpolation from finite sequences of coin tosses. Use various values of p, and try to get enough detail to allow you to get a good picture of the graph of P.

3. For most values of p, the function P is pathological, but it has many interesting properties. Is it either differentiable or continuous? anywhere? How might one attempt to define the arc length of the graph of P?