

The 18.821 Project Report

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modified by Tom Mrowka, Richard Stanley, and Haynes Miller

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The 18.821 Project Report

What a report is: A document presenting the project as you have defined it, the main findings your team has obtained, and how you obtained them.

Findings may come in many forms:

- ▶ rigorous results
- ▶ heuristic arguments
- ▶ conjectures
- ▶ observations
- ▶ data

The **target audience** is students about at the same level as yourself.

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What a report is not:

A project report is not an exposition of what's known on the subject.

It is also not a research paper (necessarily).

It's not a lab notebook.

Length: It should be long enough to convey your project and your findings, without padding or undue brevity. Generally this is around ten pages excluding possible appendices.

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Style:

Generally speaking, the 18.821 Lab Report adheres to the style of a paper in a research journal in mathematics.

Mathematics journal articles are organized as follows:

Title and Abstract

Introduction

(Background)

Body

(Appendices)

References

Here are the parts of a sample 18.821 paper, in several versions.

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Title and Abstract: First version

Enumeration of colorings of arcs of projections of a knot K

Al Dough, Bea Row, and Cee Low

The *title* should convey the subject-matter in a punchy but accurate way. Avoid symbols.

Abstract. We look at possible ways of colorings the arcs in the plane projection of a knot. We prove that this gives rise to a knot invariant, which can distinguish infinitely many different equivalence classes of knots.

The *abstract* outlines the contents of the paper in a few sentences, precisely but without definitions or explanations.

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Title and Abstract: Second version

Coloring Knots

Al Dough, Bea Row, and Cee Low

The *title* should convey the subject-matter in a punchy but accurate way. Avoid symbols.

Abstract. We describe certain colorings of the arcs of a knot projection. We prove that enumeration of these colorings gives rise to a knot invariant that can distinguish infinitely many different equivalence classes of knots.

The *abstract* outlines the contents of the paper in a few sentences, precisely but without definitions or explanations.

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The *introduction* gets the readers involved in the subject by describing the context of the problem, what the paper achieves, and what methods are used. It should end with a brief description of the structure of the paper, and acknowledgements.

Questions to ask about this section:

Does it give the reader a good intuitive grasp of what the problem is?

Does it express the authors' approach to the problem?

Does it give the structure of the paper and acknowledgements?

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1. Introduction (First version)

In this project, we investigate knots. The main question in knot theory is: *how can we tell whether a knot can be deformed into another one?* Many knot theorists have devoted a lot of time to classifying knots. Their work gave rise to the Rolfsen table of Prime knots [1]. Many invariants (the *Alexander polynomial*, the *Jones polynomial*, the *knot group*, and *knot colorings*) have been developed for this purpose.

The structure of our paper is as follows. In the next section, we explain knots and knot projections, as well as the Reidemeister moves. Section 3 introduces the notion of knot 3-coloring, and proves that 3-colorability is unchanged under Reidemeister moves. Section 4 generalizes this to n -colorability for any n . Section 5 discusses example computations. Section 6 explains how making certain modifications to a knot affects the number of 3-colorings. Sections 1–2 were written by Al Dough, and Sections 5–6 by Bea Row. Sections 4 and 7 were written by Cee Low. Section 3 was written by Row and Low.

1. Introduction (Second version)

Mathematical knots are closed loops or strings in space, without kinks or self intersections. These objects lend themselves to simple experimentation, but also have important applications.

Mitochondrial DNA is a concrete example where approximately similar structures appear in nature [1].

The main theoretical tool in studying knots are *knot invariants*. A knot invariant is a number attached to a knot, and which is unchanged under deformations. This, for instance, allows one to prove that certain knots cannot be deformed into a simple circle. Our project studies *3-colorings* of knots, which are ways of labeling segments of a knot with three possible labels. It will turn out that 3-colorability of a knot is a knot invariant, which is sufficiently strong to distinguish the first inequivalent examples of knots (the unknot and the trefoil knot). The proof that 3-colorability is an invariant uses a case-by-case analysis of Reidemeister moves.

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1. Introduction (Third version)

This paper studies knots and knot colorability. Our main result is this:

Theorem. (Theorem 4.3 in the paper) *3-colorability of a knot is invariant under Reidemeister moves.*

This means that it is an invariant of the knot. For instance, one can use this to prove that the trefoil cannot be continuously deformed into the unknot: it is genuinely knotted. In fact, we will prove the stronger result that *the number of 3-colorings is a knot invariant.*

The main strategy will be to reduce questions about 3-colorings to linear algebra, via a certain incidence matrix associated to a knot diagram. Reidemeister moves give rise to Gauss operations (row or column operations) of this matrix. This is familiar from linear algebra, with the twist that our ground field is the finite field \mathbb{F}_3 .

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1. Introduction (Fourth version)

Mathematical knots are closed loops or strings in space, without kinks or self-intersections. The main question in knot theory is: *how can we tell whether a knot can be deformed into another one?* For instance, look at the following two knots (figures taken from [3,5]):

Image of an example of a knot removed due to copyright restrictions.

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The one on the right is an *unknot*, which means that it can be deformed into a circle; in contrast, the one on the left, called a *trefoil knot*, is a genuine nontrivial knot. The first fact can be shown by going through an elementary sequence of transformations; however, the second one requires more advanced tools, namely *knot invariants*.

A knot invariant is a number (or other object) attached to a knot, which is unchanged under deformations. Our project studies *knot 3-colorings*, which are ways of labeling segments of a knot with three possible labels. It will turn out that 3-colorability of a knot is a knot invariant. This is the main result of the paper (Theorem 3.1), and will be shown using a case-by-case analysis of knot deformations. The unknot is not 3-colorable, but the trefoil is; hence, we can show that they cannot be deformed into each other (Example 4.2). Using the number of colorings as an invariant, we'll be able to find more inequivalent knots (Corollary 4.4). We conjecture (Conjecture 4.1) the stronger result that the *number* of 3-colorings is a knot invariant.

Knot colorability can be generalized from 3 colors to n , for any $n \geq 3$. We discuss this generalization in Section 5. It turns out that these are genuinely better invariants: there are knots which can be distinguished by their 5-colorings but not by their 3-colorings (Example 5.2).

Sections 1–2 were written by Al Dough, and Sections 5–6 by Bea Row. Sections 4 and 7 were written by Cee Low. Section 3 was written by Row and Low.

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Sometimes a “background section” is useful. It introduces context, definitions, and perhaps notation, in precise terms.

Questions to ask about this section:

Are the contents well motivated?

Is the level of detail and precision appropriate?

Are these items actually needed in the sequel?

Is the exposition well-organized?

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2. Background (First version)

Definition 2.1. An *oriented knot* is a smooth map $k : \mathbb{R} \rightarrow \mathbb{R}^3$ such that $k(s) = k(t)$ if and only if $s - t \in \mathbb{Z}$, as well as $k'(t) \neq 0$ for all t . An *unoriented knot* is such a map considered up to time-reversal $k(t) \leftrightarrow k(-t)$.

In this paper, we will consider only *unoriented* knots. In fact, instead of k itself we consider the image $k(\mathbb{R}) \subset \mathbb{R}^3$, which is a closed loop.

Definition 2.2. Two oriented knots k_0, k_1 are called *isotopic* if there is a family of knots k_s , smooth with respect to the parameter $s \in [0, 1]$, joining them. Unoriented knots are called isotopic if, for some choice of orientation, the associated oriented knots are isotopic. The intuitive image is that the knot k_s moves around, without acquiring self intersections or kinks. In fact, we will study knots via knot projections.

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Definition 2.3. A knot projection is a smooth map $l : \mathbb{R} \rightarrow \mathbb{R}^2$ such that $l(t) = l(t + 1)$ for all t , $l'(t) = 0$ for all t , and with at most finitely many double crossings. At each crossing, the two branches should meet transversally, and we distinguish one of them as overbranch (the other being the underbranch).

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2. Background (Second version)

Intuitively, a *mathematical knot* $K \subset \mathbb{R}^3$ can be thought of as a closed, tangled loop of string in 3-dimensional space. The formal definition is that K is the image of an infinitely differentiable map $k : \mathbb{R} \rightarrow \mathbb{R}^3$ which satisfies:

$k(t) = k(t + 1)$ for all t (this makes K into a closed circle);

$k'(t) \neq 0$ (this means that there are no kinks or singular points);

$k(s) \neq k(t)$ for all $0 \leq s < t < 1$ (this avoids self intersections).

However, we will mostly appeal to geometric intuition, avoiding formal details. Two knots are called *equivalent* if they can be deformed smoothly into each other.

Mostly, we will study knots through their projections. A knot projection is a circle drawn in the plane with only transverse self intersections. Moreover, at each intersection, one of the two branches is distinguished as lying on top of the other.

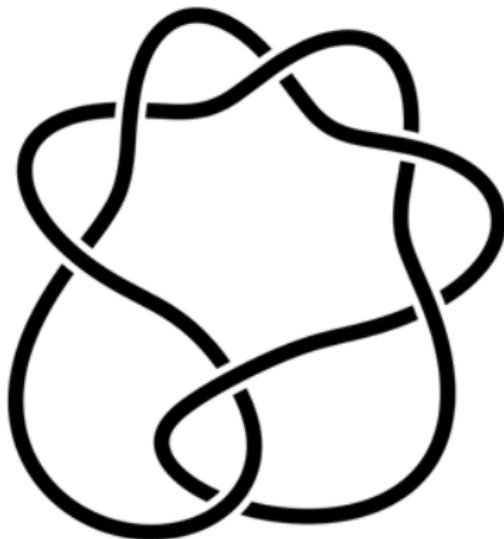
2. Background (Third version)

A *knot* K is a circle smoothly embedded in three-dimensional space. Intuitively, it can be thought of as a closed piece of string. It can be tangled, but may not have kinks or self intersections. Two knots are considered equivalent if they can be smoothly deformed into each other (again, without causing kinks or self intersections). There are precise definitions of these notions involving calculus, but we will not really need them, since all our study of knots is done via their projections to the plane.

A *knot projection* P is a closed smooth curve in the plane \mathbb{R}^2 , which has only ordinary self intersections, and with some additional information. Having ordinary self intersections means that at most two branches of the curve meet at any point.

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Moreover, the two branches which meet always cross each other transversally, meaning that they have different tangent directions. The additional information is that at any crossing point, we single out one of the branches as the overcrossing (and the other as the undercrossing). Knot projections are usually drawn like this (taken from [4]):



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Each knot projection P describes a knot $K = K(P)$, in a way which is unique up to equivalence. Different plane projections can also describe equivalent knots (for instance, in the Introduction we saw a very complicated projection whose associated knot is equivalent to the unknot, described by the circle in the plane).

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Presentation of results and supporting arguments

There is no optimal way of presenting mathematics, but there are some very definite rules and conventions.

Questions to ask:

Is the status of each statement (claimed as proven, quoted from literature, conjectured) clear?

Are successive statements adequately connected to each other?

Are arguments correct and correctly presented?

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3. Main theorem (First version)

Theorem 3.1. *3-colorability is a knot invariant.*

We will prove this by showing that it is invariant under Reidemeister moves. Type I is straightforward. Type II is still simple, but already requires the distinction between two cases (of strands with equal or unequal colors). Type III is divided into many cases, and we'll only do the most interesting one (where the colors are as distinct as possible).

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3. Main theorem (Second version)

Theorem 3.1. *3-colorability is a knot invariant.*

Our approach is slightly indirect, and involves the arithmetic of the field \mathbb{F}_3 of integers modulo 3. Given a nontrivial knot projection P with n crossings, we define an n by n matrix $A = A(P)$ with coefficients in \mathbb{F}_3 , as follows. Columns of the matrix are labeled by arcs, and rows are labeled by crossings. The (i, j) -th entry A_{ij} is 1 if the j -th crossing lies on the i -th arc (which can mean that it is an overcrossing, or that it is an undercrossing and the arc ends there). All other entries are set to be 0. We will show first the following:

Lemma 3.2. *A vector $v \in (\mathbb{F}_3)^n$ is a coloring of P if and only if $Av = 0$.*

This includes the trivial colorings $v = (i, \dots, i)$, which always exist and are ruled out by our usual terminology. As an easy consequence, we get:

Lemma 3.3. *The knot projection P is colorable if and only if $\text{rank}(A) < n - 1$.*

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Presentation of results and supporting arguments

Three weeks isn't very much time, and mathematics is more than formal statement and proof. Often you will want to present observations or conjectures.

Questions to ask:

Is the status of each statement (claimed as proven, quoted from literature, conjectured) clear?

Are successive statements adequately connected to each other?

Are arguments correct and correctly presented?

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4. The number of colorings (First version)

The result about 3-colorability can be strengthened as follows:

Experimental Fact 4.1. *The number of 3-colorings is a knot invariant.*

We have no rigorous proof of this. It is obviously true for unknots, since then the number of 3-colorings is always zero. Next, we have considered 20 different projections of the trefoil (Appendix 1).

They all turn out to have the same number of colorings, supporting our conjecture. It is still possible that this is a special property of the trefoil, and fails for more complicated knots.

Experimental Fact 4.2. *The number of 3-colorings of a given knot projection is always of the form $3^n - 3$, for some integer n .* We have taken the first 12 knots from the classical knot tables [7], and computed their numbers of colorings by hand. The result of the computation (see Appendix 2) supports Experimental Fact 4.2. We have actually found a general proof in the literature [4], but it uses algebra arguments which are quite different from the ones used here.

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4. The number of colorings (Second version)

3-colorability as a knot invariant has limited power, since it can at most divide all possible knots into two classes. We would like to refine it as follows:

Conjecture 4.1. *The number of 3-colorings is a knot invariant.*

The natural way to approach this is by inspection of the proof of Theorem 3.1. Under Reidemeister moves of type I and II, that proof provides a simple bijection between colorings of the original knots and of the modified one, which provides partial evidence for the conjecture. Reidemeister III is more complicated, and we could not complete the argument in all cases.

Conjecture 4.2. *The number of 3-colorings of a given knot projection is always of the form $3^n - 3$, for some integer n .*

$3^n - 3$ is always even and divisible by three (by computations modulo 2 and 3, respectively). Note that we can permute the colors in any 3-coloring, which means that the number of colorings is always divisible by 6. This provides partial evidence for the conjecture. One could of course also get experimental evidence by looking at the classical knot tables [7].

4. The number of colorings (Third version)

In addition to the basic question of whether a knot is colorable, one can also look at the number of possible colorings.

Empirical Fact 4.1. *The number of 3-colorings is a knot invariant.*

Empirical Fact 4.2. *The number of 3-colorings is always of the form $3^n - 3$, for some integer n .*

The first fact is important since it yields an invariant that can have potentially infinitely many different values. The second fact restricts the range of this invariant. We cannot prove either statement rigorously, but there is partial evidence of various kinds: *Theoretical evidence.* The number of 3-colorings is invariant under Reidemeister moves of type I and II. This can be seen by inspecting our proof of Theorem 3.1. Namely, if two knots are related by such a move, the argument from the proof provides bijection between their possible 3-colorings. If this argument could be extended to type III, it would provide a complete proof of Experimental Fact 4.1.

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Experimental evidence. We have looked at 10 pairs of different projections of knots, differing from each other by a type III move. In all cases, the number of 3-colorings is the same before and after the move (Appendix 1), as predicted by Experimental Fact 4.1. We have also taken the first 12 knots from the classical knot tables [7], and computed their numbers of colorings by hand. The result of the computation (see Appendix 2) supports Experimental Fact 4.2.

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Citations: You need to *cite all references* (from books, articles, webpages, personal communication) that you consulted and which had an impact on your report.

Questions to ask about citations:

Does the reader know when you are quoting a result, and where you got it?

Can the reader see clearly where your original contribution lies?

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Examples of poor citation:

Useful probability textbooks are [1,2,15]...

We will use without further comment results from [5]...

Inspired by [3], we introduce the following matrix...

[5, Theorem 7] then yields the desired result...

The following argument is partly borrowed from [3] and partly original...

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Examples of good practice:

Knots will be listed in the notation from [5, pages 210-218]...

We apply the Central Limit Theorem [5, Theorem 15.7] to our probability distribution P , and conclude that...

Recall the main theorem about Reidemeister moves:

[4, Theorem 11]. *Let two knot . . .*

We take the following definition from [3, pages 10–12], which treats only the case $n = 3$, and generalize it to all n .

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Final comments:

Revision is important: Almost everyone writes and rewrites several times.

Team feedback is important: if your teammates find your text hard to read, so will others. All team members are responsible for the coherence of the document as a whole.

The first draft should be your best shot at a final draft. Don't expect us to do your laundry for you.

On the other hand, you will have more to do (maybe even more mathematics) in response to comments on the first draft.

Practice in writing mathematics will contribute important elements to your general writing skill (precision of expression, organizing a complex argument, avoiding BS).

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