

# THE DYNAMICS OF SUCCESSIVE DIFFERENCES OVER $\mathbb{Z}$ AND $\mathbb{R}$

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ABSTRACT. The  $n$ -value game is a dynamical system defined by a method of iterated differences. In this paper, we examine the behavior of several variants of the  $n$ -value game, and prove a few key results: the 4-value game over the positive integers is guaranteed to converge to a fixed point; the 3-value game over the positive integers is guaranteed to exhibit cyclic behavior; for all  $n$ , there exist infinitely many non-cycling  $n$ -value games over the positive reals with infinite length.

## 1. INTRODUCTION

The  $n$ -value game is a system based on a simple transition rule – we describe the  $n = 4$  case as an example, with other polygons generalizing naturally. Begin with a square and label its vertices with numbers  $(a, b, c, d)$  chosen from a set equipped with a norm – in this paper, we examine the sets  $\mathbb{Z}^+$  (positive integers) and  $\mathbb{R}^+$  (positive reals) with the standard absolute value norm. To apply a single step of the transition, for each edge of the square, write the absolute value of the difference between the two endpoints on the midpoint of the edge. Finally, connect these midpoints to form a new square.

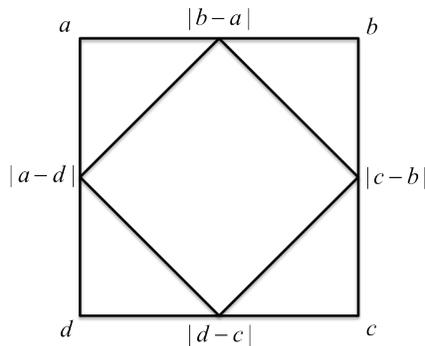


FIGURE 1. A visualization of the 4-game transition.

Specifically, for  $n = 4$ , the operator

$$T \equiv (a, b, c, d) \rightarrow (|b - a|, |c - b|, |d - c|, |a - d|)$$

represents this transformation. During an  $n$ -value game, we define each individual ordered list of vertex labels as a *state*, and the ordered sequence of states forms the *game*. The state  $s_m$  obtained by  $m$  applications of transition rule  $T$  to a starting state  $s_0$  will be notated by  $s_m = T^m s_0$ .

**Definition 1.1** (*k*-cyclic games). *A game starting from state  $s_0$  is k-cyclic iff  $\exists$  integers  $m, k > 0$  such that  $T^m s_0 = T^{m+k} s_0$ .*

Note that a game can be 1-cyclic; this corresponds to convergence towards a fixed point.

**Definition 1.2** (Length of a game). *For 1-cyclic games starting at state  $s_0$ , the length of the game is the least integer  $m$  such that  $T^m s_0 = T^{m+1} s_0$ .*

**Definition 1.3** (Non-repeating games). *A game starting from state  $s_0$  is non-repeating iff  $T^m s_0 \neq T^{m+1} s_0$  for all integers  $m \geq 0$ .*

In this paper, we prove key properties of  $n$ -value games over different sets. Section 2, authored by Matt Redmond, examines the notion of “equivalence” between states, and demonstrates that games over the rational numbers can be reduced to equivalent games over the integers. Section 3, authored by Yida Gao, discusses the behavior of all 3-value games over  $\mathbb{Z}^+$ , concluding that each 3-value game ends with 3-cyclic behavior. Section 4, authored by Matt Redmond and Zach Steward, proves that each 4-value game on  $\mathbb{Z}^+$  ends with 1-cyclic behavior, converging to the fixed point  $(0, 0, 0, 0)$ . This section also discusses empirical results and probability distributions on lengths for the 4-value game on  $\mathbb{Z}^+$  with entries taken from  $[0, v - 1]$  for various  $v$ . Section 5, authored by Matt Redmond, demonstrates the existence of non-repeating 4-value games over the reals. Furthermore, this section demonstrates the existence of infinitely many non-repeating games by utilizing the notion of equivalence developed in Section 2.

## 2. AN EQUIVALENCE RELATION ON STATES

It will be helpful to have a conception about state equivalence: namely, what should we mean by equivalent? We would like games that begin with any two “equivalent” states to have identical long-term behavior (either  $k$ -cyclic or non-repeating), with the same length if  $k = 1$ .

**2.1. Symmetries.** We begin by noting that the state  $(a, b, c, d)$  is clearly equivalent to the state  $(b, c, d, a)$  - all of the vertex labels have been rotated one place to the left. Similarly, the state  $(a, b, c, d)$  is also equivalent to the state  $(b, a, d, c)$  - the vertex labels have been reflected across a vertical line. The diagram below captures all of the symmetries of the square. These symmetries form the group  $D_8$  - the dihedral symmetry group of the square.

**Definition 2.1** (Equivalent by symmetry). *For an  $n$ -value game, two states  $X$  and  $Y$  are equivalent if there exists an element  $\sigma \in D_{2n}$  such that  $\sigma(X) = Y$ .*

These symmetries  $\sigma$  preserve vertex adjacency, so they will preserve our transformation rule, which depends only on adjacent vertices.

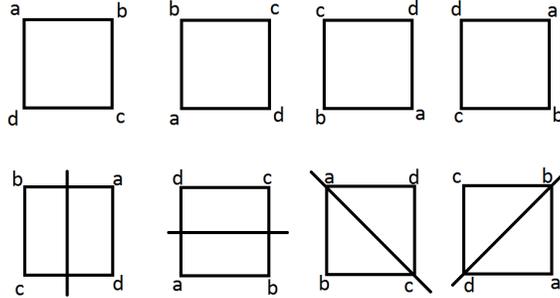


FIGURE 2. Elements of  $D_8$ , the symmetry group of the square, with rotations on top and reflections on bottom.

**2.2. Scalar multiplication.** States that are a constant multiple of each other have identical long-term behavior - this can also be used to help us define equivalence.

**Lemma 2.2.** *If  $r \in \mathbb{R}^+$ , the game starting with  $(ra_1, ra_2, \dots, ra_n)$  has the same behavior as the game starting with  $(a_1, a_2, \dots, a_n)$ .*

*Proof.* Let  $p = (ra_1, ra_2, \dots, ra_n)$ , and  $q = (a_1, a_2, \dots, a_n)$ . By the linearity of subtraction (and the fact that  $r > 0$  by hypothesis), we have  $Tp = (|ra_2 - ra_1|, |ra_3 - ra_2|, \dots, |ra_1 - ra_n|) = (r|a_2 - a_1|, r|a_3 - a_2|, \dots, r|a_1 - a_n|) = r \cdot Tq$ . The steps taken by the game starting with  $q$  are therefore exactly the same steps taken by the game starting with  $p$ , with an extra multiplier of  $r$  pulled out in one step, which will not alter the long term behavior.  $\square$

**Definition 2.3** (Equivalent by scaling). *Two states  $X$  and  $Y$  are equivalent if  $\exists r \in \mathbb{R}^+$  such that  $r \cdot X = Y$ .*

**2.3. Constant offset.** States that differ by a constant offset will clearly have long term behavior that is identical, because the offset is absorbed after one transition.

**Lemma 2.4.** *If  $k \in \mathbb{R}$ , the game starting with  $(a_1, a_2, \dots, a_n)$  has the same behavior as the game starting with  $(a_1 + k, a_2 + k, \dots, a_n + k)$ .*

*Proof.* Consider  $a = (a_1 + k, a_2 + k, \dots, a_n + k) = a + k$  for some constant vector  $k$ .  $Ta = (|((a_2 + k) - (a_1 + k))|, |((a_3 + k) - (a_2 + k))|, \dots, |((a_n + k) - (a_1 + k))|) = (|a_2 - a_1|, |a_3 - a_2|, \dots, |a_n - a_1|) = Ta$ . Applying  $T$  on some starting vector plus an offset yields the same result as applying the transform to the starting vector:  $T(a + k) = Ta$ .  $\square$

**Definition 2.5** (Equivalent by offset). *Two states  $X$  and  $Y$  are equivalent if  $\exists k \in \mathbb{R}$  such that  $X + (k, k, \dots, k) = Y$ .*

**2.4. Equivalence.** Putting together the previous notions of equivalence, we arrive at a combined definition for our equivalence relation:

**Definition 2.6** (Equivalence of states). *For an  $n$ -value game, two states  $X$  and  $Y$  are equivalent if  $\exists r, k \in \mathbb{R}$  with  $r > 0$ , and an element  $\sigma$  of  $D_{2n}$  such that  $r \cdot \sigma(X) + (k, k, \dots, k) = Y$ .*

This notion of equivalence under scaling, offset, and symmetry will prove useful for later discussion.

**2.5. Games over  $\mathbb{Q}$  and  $\mathbb{Z}$ .** It is tempting to consider the behavior of  $n$ -value games over the rationals instead of the integers; however, this is a fruitless endeavour.

**Corollary 2.7** (States over  $\mathbb{Q}$  reduce to states over  $\mathbb{Z}$ ). *States with all entries in  $\mathbb{Q}$  are equivalent to a state with all entries in  $\mathbb{Z}$ .*

*Proof.* Suppose we have a state  $s$  whose entries are rational numbers:  $s = (\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n})$ . We can rewrite  $s$  with a common denominator  $D = \prod b_i$ :

$$s = \left( \frac{a_1 b_2 b_3 \cdots b_n}{D}, \frac{a_2 b_1 b_3 \cdots b_n}{D}, \dots, \frac{a_n b_1 b_2 \cdots b_{n-1}}{D} \right).$$

From 2.2, we can pull out the constant scalar  $\frac{1}{D}$  and show that this is equivalent to the state

$$(a_1 b_2 b_3 \cdots b_n, a_2 b_1 b_3 \cdots b_n, \dots, a_n b_1 b_2 \cdots b_{n-1})$$

which has all integer entries. This demonstrates that all states over the rationals can be reduced to an equivalent state over the integers.  $\square$

3. BEHAVIOR OF ALL 3-VALUE GAMES OVER  $\mathbb{Z}^+$

In this section, we consider the behavior of the 3-value game over  $\mathbb{Z}^+$ . We prove that all 3-value games with a non-trivial initial state cycle, rather than converging to  $(0,0,0)$ .

First, let us imagine the values in the tuple as points on a number line. For example, a starting triangle with  $(1,3,5)$  looks like this on the number line:



FIGURE 3. A number line with points corresponding to  $(1,3,5)$  game state.

**Definition 3.1.** Let  $range(s)$  be defined as the largest positive difference between any two points in a given state  $s$ .

$$range(s) = \max_{s_i \in s} (s_i) - \min_{s_i \in s} (s_i)$$

**Definition 3.2.** A *trivial 3-value game* is one in which the start state is  $(x, x, x)$ , where  $x \in \mathbb{Z}^+$ .

Thus, a trivial 3-value game with start state  $s_0$  applied with transition operator  $T$  leads to  $Ts_0 = (0, 0, 0)$  on the first step.

**Definition 3.3.** A *non-trivial 3-value game* is one in which  $Ts_0 = (0, 0, 0)$  where  $T$  is the transition operator and  $s_0$  is the start state of the game with values  $\in \mathbb{Z}^+$ .

**Theorem 3.4.** All non-trivial 3-value games over  $\mathbb{Z}^+$  enter a 3-cycle of the form  $(0, x, x)$ .

*Proof.* The proof is by cases. Because adjacency of the vertices will be preserved for any  $\sigma \in D_6$  we only need to consider five possible cases for the non-trivial 3-value game over  $\mathbb{Z}$ . The cases are states of 3-value games prior to a transition. The first three cases immediately lead to a 3 step cycle while the last two cases lead to decreases in range.

- (1) One zero and two numbers of the same value  $(0, x, x)$ :

This case enters a 3-cycle that returns a permutation of  $(0, x, x)$  on every step.

$$(0, x, x) \rightarrow (|0 - x|, |x - x|, |x - 0|) = (x, 0, x)$$

$$(x, 0, x) \rightarrow (|x - 0|, |0 - x|, |x - x|) = (x, x, 0)$$

$$(x, x, 0) \rightarrow (|x - x|, |x - 0|, |0 - x|) = (0, x, x)$$

- (2) Two zeros and one non-zero number  $(0, 0, x)$ :

Range stays the same and the game enters *case 1*.

$$(0, 0, x) \rightarrow (0 - 0, |0 - x|, |x - 0|) = (0, x, x)$$

Range of  $(0, 0, x) = x$ ; range of  $(0, x, x) = x$

- (3) Three non-zero values in which two values are the same  $(x, y, y)$ :

Range stays the same and the game transitions to *case 1*.

$$(x, y, y) \rightarrow (|x - y|, 0, |y - x|)$$

Range of  $(x, y, y) = |x - y|$ ; range of  $(|x - y|, 0, |y - x|) = |x - y|$

- (4) Three distinct non-zero values  $(x, y, z)$ :

Since with the 3-value game we are essentially dealing with three numbers on a number line, any permutation of a state we want can be produced with a simple rotation of the values. Thus, without loss of generality, let  $z > y > x$ . In this case, the range decreases by  $y - x$  or  $z - y$ , and the new range is  $z - y$  or  $y - x$ .

$$(x, y, z) \rightarrow (|x - y|, |y - z|, |z - x|) = (y - x, z - y, z - x)$$

If  $z - y > y - x$ , new range =  $z - x - (y - x) = z - y$ , otherwise new range =  $z - x - (z - y) = y - x$ . The difference in range is either  $z - x - (z - y) = y - x$  or  $z - x - (y - x) = z - y$ .

Since the three distinct values case always implies reduction of range, the game will always reach a state with a repeated entry, which will then step into the cycling *case 1*.

- (5) One zero and two numbers of different values  $(0, x, y)$ :

In this case, the range decreases by  $y - \min\{y - x, x\}$ . Without loss of generality, assume  $0 < x < y$ :

$$(0, x, y) \rightarrow (|0 - x|, |x - y|, |y - 0|) = (x, y - x, y)$$

Range of  $(0, x, y) = |y - 0| = y$ ; range of  $(x, |x - y|, y) = y - (y - x) = x$  or  $y - x$ . Thus the range decreases by  $y - x$  or  $y - (y - x) = x$ .

For all non-trivial 3-value games, the range is guaranteed to either 1. stay the same, in which case the game transitions to cycling *case 1* or 2. decrease at each step so the game will eventually transition into a  $(x, y, y)$  (*case 4*) or  $(0, 0, x)$  (*case 3*) state, which both lead to the *case 1* state. Thus, all non-trivial 3-value games over  $\mathbb{Z}$  will reduce to *case 1* and cycle.  $\square$



*Proof.* From any starting state  $g = (a_1, a_2, a_3, a_4)$  with all entries in  $\mathbb{Z}^+$ , take several steps until the game reaches a state where all entries are even. This will take at most four steps, by 4.1. The new state  $g^{\text{even}}$  can be written as  $(2b_1, 2b_2, 2b_3, 2b_4)$ . By 2.2, the length of  $g^{\text{even}}$  is exactly the same as the length of the game starting on  $(b_1, b_2, b_3, b_4)$ . However, we are guaranteed that the maximum element in  $(b_1, b_2, b_3, b_4)$  has decreased from the maximum element in  $(a_1, a_2, a_3, a_4)$  by at least a factor of 2.

Proceed inductively, by stepping each new game until all entries are even (at most four steps each time), then factor out another 2. As the maximum element is constantly decreasing, each game must terminate at  $(0, 0, 0, 0)$  in a finite number of steps. Note that this last claim requires the well-ordering property: there is a least non-negative integer (namely, zero) that we terminate at. As we will see in Section 5, the real numbers do not have this property, and we cannot bound the game length in the same way.  $\square$

**Theorem 4.3.** *Let  $L$  be the length of a game starting on  $(a, b, c, d)$ . Then  $L < 4\lceil \log_2(\max(a, b, c, d)) \rceil$*

*Proof.* Each iteration of our induction requires at most four steps to reach a state with all even entries, and then it removes a factor of two from the state. We can divide by two at most  $\lceil \log_2(\max(a, b, c, d)) \rceil$  times from the state  $(a, b, c, d)$  before terminating; therefore, the length is at most four times as large as the number of times we can pull out a factor of two.  $\square$

**4.2. The distribution of game lengths for the 4-value game on subsets of the integers.** After demonstrating that all 4-value games over  $\mathbb{Z}^+$  end at a fixed point, it is clear that a program which computes the length of various 4-games over  $\mathbb{Z}^+$  will definitely halt for all inputs. We wrote such a program that takes a parameter  $v$  and examines all possible games with integer entries between 0 and  $v - 1$ , and used it to make a few empirical observations about the probability distribution on game length. In this section, we will examine this probability distribution in order to gain a better understanding of the dynamics of the 4-game over  $\mathbb{Z}^+$ . In addition to computing statistics about measures of central tendency, we also examine extremal statistics: what is the maximum length for games with entries in  $[0, v - 1]$  for some parameter  $v$ ? How does this observed maximum length compare with the bound on length given in 4.3?

4.2.1. *Probability densities.* Figure 5 plots the probability distribution on game length for  $v = 64, 128, 256$  to demonstrate the very close match these distributions have for increasing values of  $v$ . It seems to be the case that the generally bimodal shape of this distribution does not depend on the value of the  $v$  parameter, although the maximum value clearly does.

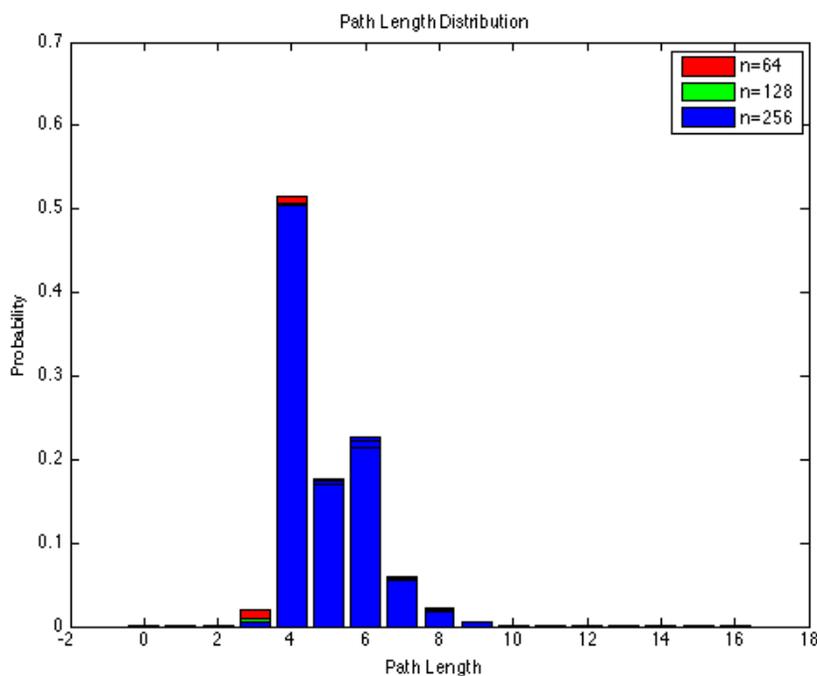


FIGURE 5. Probability Distribution of Length for  $v = \{64, 128, 256\}$

As can be seen in Figure 6, a large number of games converge to the final state  $(0, 0, 0, 0)$  after just 4 steps - cumulatively, more than 50% of these games terminate in 4 or fewer steps, and 91% terminate in 6 or fewer steps. We conjecture this is due to the fact that many games will fall into a particular equivalence class (see Section 2) that has game length 4.

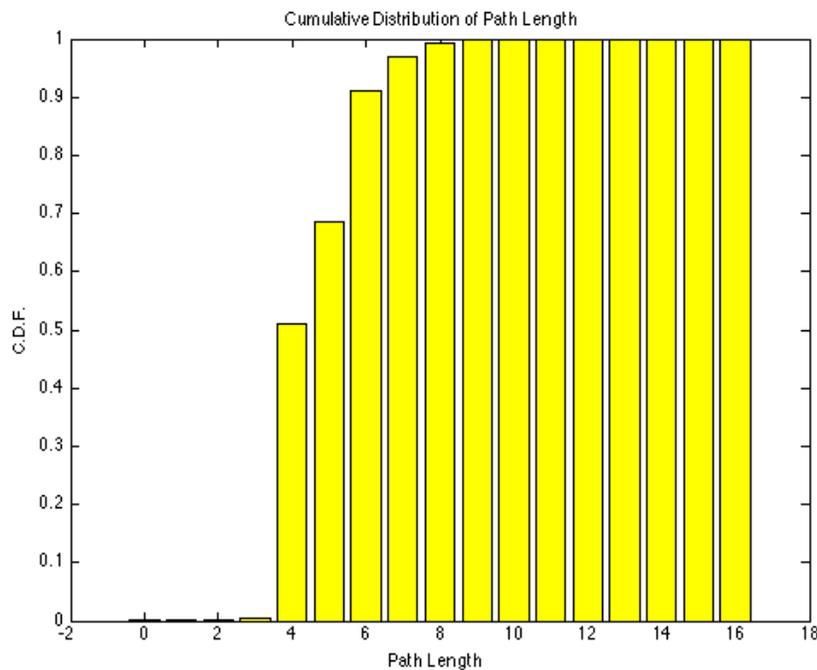


FIGURE 6. Cumulative Distribution of Length for  $v = 256$

4.2.2. *Theoretical bound for specified length.* In 4.3 we prove that the length  $L$  of a game  $(a, b, c, d)$  can be at most  $4\lceil\log_2(\max(a, b, c, d))\rceil$ , but we would like to investigate just how tight of a bound this really is. First note that for  $v = 128$  we have at best  $\max(a, b, c, d) = 127$  and therefore have a length  $L$  at most  $4 \cdot \log_2 127 = 28$ , but we are observing a maximum length of only 15. Similarly, for  $v = 64$  we observe a maximum length of 13 compared to a theoretical max of 24. Furthermore when we increase  $n$  to 256, we observe a maximum length of 16 compared to the theoretical max of 32. The reasons for this are unclear - we conjecture that this surprising result is due to the fact that, on average, the parity will appear as  $(e, o, e, o)$  or  $(e, e, o, o)$ , which only take 2 or 3 steps to reach  $(e, e, e, e)$  respectively instead of the worst-case 4 steps from the bound. In summary, the bound is quite loose.

5.  $n$ -VALUE GAMES OVER  $\mathbb{R}$

In this section, we consider the properties of the  $n$ -valued game over the real numbers. Several questions come to mind: do all real-valued games terminate at a fixed point? If not, does there exist a real-valued game that demonstrates cyclic behavior? If not, does there exist a non-repeating real-valued game of infinite length? While we cannot yet answer the second question, we answer the first question (no) and third question (yes) by proving the existence of infinitely many games with infinite length. We accomplish this by representing a single step of the game as a linear operator (with a restricted domain), then demonstrating the existence of an infinite game for each value of  $n$ . Finally, we show that every infinite length game can be modified to generate infinitely many games of infinite length by using our equivalence relation.

**Theorem 5.1** (Existence of non-repeating  $n$ -value game). *For every  $n \geq 3$ , there exists a  $\lambda_n$  such that*

$$\begin{bmatrix} (1 - \lambda_n) \\ (1 - \lambda_n)(1 + \lambda_n) \\ (1 - \lambda_n)(1 + \lambda_n)^2 \\ \vdots \\ (1 - \lambda_n)(1 + \lambda_n)^{n-2} \\ 1 \end{bmatrix}^T$$

*is a starting state for a game over  $\mathbb{R}^+$  that exhibits non-repeating behavior.*

In the following sections, we identify the value of  $\lambda_n$  as the positive real root of the equation  $(1 - \lambda_n)(1 + \lambda_n)^{n-1} = 1$ , and provide a construction of the particular state above.

**5.1. Linearizing the  $n$ -value game.** Given an  $n$ -value game on  $\mathbb{R}$ , with start state  $g = (a_1, a_2, \dots, a_n)$ , we produce each step by the transformation rule  $g_t \rightarrow g_{t+1} = (a_1, a_2, \dots, a_n) \rightarrow (|a_2 - a_1|, |a_3 - a_2|, \dots, |a_1 - a_n|)$ . We can eliminate the use of the absolute value function by restricting the domain of the input to the set of vectors  $(m_1, m_2, \dots, m_n)$  such that  $m_1 < m_2 < \dots < m_n$ . With this “increasing order” constraint, we can write  $g_t \rightarrow g_{t+1}$  as an  $n \times n$  linear operator  $T_n$ :

$$T_n = \begin{bmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & -1 & 1 \\ -1 & 0 & \dots & \dots & 0 & 1 \end{bmatrix}.$$

To compute the next state in a game, left-multiply the current state by  $T_n$ . As an example, consider the effects of  $T_4$  on  $g = (1, 5, 7, 11)^T$ :

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 4 \\ 10 \end{bmatrix}.$$

As this example shows, it is not necessarily the case that the output  $g_{t+1}$  maintains the “increasing order” constraint. In general, increasing inputs are not guaranteed to produce increasing outputs. If  $\mathbf{g}_t$  happens to be a positive, increasing eigenvector of  $T_n$  (with a positive eigenvalue), however, we *are* guaranteed that the invariant will hold: the output  $\mathbf{g}_{t+1}$  is guaranteed to be a scalar multiple of  $\mathbf{g}_t$  because  $T\mathbf{g}_t = \lambda\mathbf{g}_t = \mathbf{g}_{t+1}$ .

If our initial state  $\mathbf{g}$  is a real non-zero eigenvector of  $T_n$ , then we are guaranteed that  $T_n\mathbf{g} = \lambda\mathbf{g} = 0$ . In general, for all  $k$ ,  $T_n^k\mathbf{g} = \lambda^k\mathbf{g} = 0$ , so real, increasing eigenvectors of  $T_n$  are guaranteed to generate infinite length games.

To demonstrate that there exists an infinite length game for all  $n$ , we must demonstrate the existence of a real, increasing, nonzero eigenvector/value pair  $\mathbf{v}_n, \lambda_n$  for all  $n$ .

**5.2. Establishing and bounding a positive real eigenvalue.** We seek to establish the existence of such a positive, increasing eigenvector/eigenvalue pair. To do so, we look for the roots of the characteristic polynomial of  $T_n$ :

$$S_n = T_n - \lambda I_n = \begin{bmatrix} -1 - \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 - \lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & -1 - \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & -1 - \lambda & 1 \\ -1 & 0 & \dots & \dots & 0 & 1 - \lambda \end{bmatrix}.$$

Expanding  $\det(S_n)$  by cofactors along the bottom row reduces the sub-determinants to two triangular matrices with special forms.

$$\det(S_n) = -1(-1)^{1+n} \det \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1-\lambda & 1 & 0 & \dots & 0 \\ 0 & -1-\lambda & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1-\lambda & 1 \end{pmatrix} +$$

$$(1-\lambda)(-1)^{n+n} \det \begin{pmatrix} -1-\lambda & 1 & 0 & \dots & 0 \\ 0 & -1-\lambda & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & -1-\lambda \end{pmatrix}.$$

The determinant in the first term reduces to 1, and the determinant in the second term reduces to  $(-1-\lambda)^{n-1}$ . The characteristic polynomial of  $T_n$  is  $(-1)^{2+n} + (1-\lambda)(-1)^{2n}(-1-\lambda)^{n-1}$ . Thus, the characteristic equation is

$$(*) \quad (-1)^{2+n} + (-1-\lambda)^{n-1} - \lambda(-1-\lambda)^{n-1} = 0$$

or

$$(**) \quad (1-\lambda)(1+\lambda)^{n-1} = 1$$

We examine the pattern of signs on this polynomial to determine the number of positive roots. In each case,  $\lambda = 0$  is a root, so the coefficient on the constant term is zero.

When  $n$  is even, the sign pattern is  $(\underbrace{+, \dots, +}_{\frac{n}{2}}, 0, \underbrace{-, \dots, -}_{\frac{n}{2}-1}, 0)$ .

When  $n$  is odd, the sign pattern is  $(\underbrace{-, \dots, -}_{\frac{n+1}{2}}, \underbrace{+, \dots, +}_{\frac{n-1}{2}}, 0)$ .

Each case has exactly one change of sign, so there exists exactly one positive real root  $0 < \lambda_n$  for each characteristic polynomial by Descartes' Rule of Signs [1]. Additionally, we claim that  $\lambda_n < 1$  for all  $n$ : dividing a polynomial  $P(x)$  by  $(x-k)$  will result in a polynomial with all positive coefficients if  $k$  is an upper bound for the positive roots [2, Eqn. 15]. Dividing each of the characteristic polynomials by  $(\lambda_n - 1)$  - easily accomplished by using the form from equation (\*\*) - yields polynomials with all positive coefficients for all  $n$ , which demonstrates that 1 is always the least integral upper bound.

**5.3. Identifying an increasing eigenvector.** To determine the corresponding eigenvector  $\mathbf{v}_n = (a_1, a_2, \dots, a_n)$ , we solve  $(T_n - \lambda_n I_n)\mathbf{v}_n = \mathbf{0}$ . This produces the following set of equations:

$$\begin{array}{ccc} (-1 - \lambda_n)a_1 + a_2 = 0 & & (1 + \lambda_n)a_1 = a_2 \\ (-1 - \lambda_n)a_2 + a_3 = 0 & & (1 + \lambda_n)a_2 = a_3 \\ & \vdots & \\ & & \text{or} & & \vdots & \\ (-1 - \lambda_n)a_{n-1} + a_n = 0 & & (1 + \lambda_n)a_{n-1} = a_n \\ (1 - \lambda_n)a_n - a_1 = 0 & & (1 - \lambda_n)a_n = a_1 \end{array}$$

Arbitrarily, let  $a_n = 1$ . This forces  $a_1 = (1 - \lambda_n)$ , which forces  $a_2 = (1 - \lambda_n)(1 + \lambda_n)$ . In general, for  $1 \leq i < n$  we have  $a_i = (1 - \lambda_n)(1 + \lambda_n)^{i-1}$ . An eigenvector that corresponds to the eigenvalue  $\lambda_n$  is thus

$$\begin{bmatrix} (1 - \lambda_n) \\ (1 - \lambda_n)(1 + \lambda_n) \\ (1 - \lambda_n)(1 + \lambda_n)^2 \\ \vdots \\ (1 - \lambda_n)(1 + \lambda_n)^{n-2} \\ 1 \end{bmatrix}.$$

We verify that the entries of this eigenvector are in increasing order for all  $n$ : we have  $(1 - \lambda_n)(1 + \lambda_n)^k < (1 - \lambda_n)(1 + \lambda_n)^{k+1}$  because  $(1 + \lambda_n)^k < (1 + \lambda_n)^{k+1}$  when  $0 < \lambda_n$  and  $(1 - \lambda_n) > 0$  when  $\lambda_n < 1$  - we can claim  $\lambda_n < 1$  from the bound established above, and indeed, we conjecture that  $\lim_{n \rightarrow \infty} \lambda_n = 1$ .

Empirically, for the  $n = 4$  case, we have  $\lambda_4 \approx 0.839287$ , so the eigenvector which generates a game of infinite length is approximately  $g = (0.160713, 0.295598, 0.543689, 1)$ . The progression of this game after  $t$  timesteps results in  $g_t = (0.839287)^t \cdot (0.160713, 0.295598, 0.543689, 1)$ .

#### 5.4. Generating infinitely many solutions of infinite length.

Our choice of  $a_n = 1$  was arbitrary - the eigenvector we obtained was parametrized only on  $a_n$ . Choosing other values of  $a_n > 1$  will lead to infinitely many such solutions. Additionally, we can use any other equivalent starting state to generate a new non-repeating game.

## REFERENCES

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