

Comments on "The n-value game over Z and R"

Introduction. This is not well organized. "from a polygon..." is incomplete, and repeated later, for example. You elect to describe the $n=4$ case, which seems like a good choice. But just do that right off, completely.

You can't say "repeat until all vertices are zero," because, as you show, this doesn't always happen. This makes your notion of "length" ill defined. (Also, I think some people would prefer to say that what you call an infinite game is a non-repeating game.) You might come up with a rough breakdown of the dynamics: any game is either non-repeating, or eventually gets stuck in a k -cycle. If $k = 1$ this is a fixed point. Games which eventually become constant have a "length."

State your results in the introduction. You may not want to state them precisely (e.g. the exact enumeration may be out of place there), but give some precise statements. You will want to say what you mean by "n-value game over A" where A is either Z or R. (Come to think of it, what you really need is a set S with a function $S \times S \rightarrow S$, written with absolute values. It might have been fun to see what sort of axioms on $|\cdot|$ make for interesting dynamics. You are actually using this generalization when you talk about "games over the integers in $[0, n-1]$."))

Then say where the different topics are studied, section by section.

Finally give the breakdown of authorship.

Since any game over Z or R immediately transitions to a game over N or R_+ (natural numbers, nonnegative reals; N is standard notation but R_+ needs definition), there's a good argument for thinking of them as defined over these subsets and not over all of Z or R.

Several times in the paper you use symmetry arguments. In the 3-game case, you use the symmetric group on 3 letters and its action on N^3 by permuting the coordinates. What is the exact statement here? It is not true that $T \circ \sigma = \sigma \circ T$. (Here I use T for the self-map describing your dynamical system.) For example, in case (5) on p 4, why can you assume that $z > y > x$? In any case, the end of section 3 is the wrong place for this discussion. It might deserve a separate section. Wherever it is, say what you mean by "equivalent" games; this is the language to use in Section 4. Section 4 doesn't deal with the dynamics of these games at all; it's enumerating the orbits of the D_8 action on $[0, n-1]^4$. But the reason it is interesting in the context of this paper is that vectors in the same orbit lead to "equivalent" dynamics.

Def 2.6 isn't really what you mean. You're really interested in where a game ends, not where it starts. I think you should first observe that if $Tg = g$ then $g = (0,0,0)$, and that if there is g such that $Tg = (0,0,0)$ then $g = (x,x,x)$. This is the "trivial" orbit. Next, there are 3-cycles: $(x,x,0)$ for $x > 0$ is part of a 3-cycle involving $(x,0,x)$ and $(0,x,x)$.

Then your theorem says that every nontrivial game eventually enters one of these 3-cycles. (I think it's interesting to ask what x is, in terms of the initial position.)

Proof of Theorem 2.7. There are two ideas here: decrease range if possible, and enter one of the standard 3-cycles. I think you should separate which applies more clearly. I think you can enlarge Case 5 to include case 2; so 3 distinct (nonnegative) entries always imply a reduction of range. (You need to fix the proof of this.) Thus you will eventually have a repeated entry, and then at the next step you enter the 3-cycle.

On 2.3: These proofs are the same as each other. What you are doing is showing that for any game (of any size) over Q , there is a game over Z with the same behavior (in the sense sketched above - non-repeating, or hitting a k -cycle for the same k).

On Section 3. This is a running account of the exploration. For a paper, after the nice introduction, it should say: Theorem. Let $n \geq 3$. There is a positive real number λ_n such that the following n -games never repeat. ... and then list the starting points. You'll have a further free parameter. I'd like to see explicitly the n -games you get from the D_8 action. As a remark following the theorem you can say that in fact λ_n is the unique positive root of the equation $(1-\lambda)(1+\lambda)^{n-1}=1$.

Section 5: This is too unfocused. I think I know how this evolved: we were discussing distribution of game length, and wondering whether it stabilized in some way as n grows large. So you have to know what you are enumerating. You seem to focus in 5.1 on the question of the proportion of starting positions with four distinct entries. I don't see the relevance to the question that is taken up later in \S5. What 5.1 seems to do is to compare the frequency of vectors with pairwise distinct entries with the frequency of equivalence classes of such vectors. But for the latter wouldn't you want to be taking the ratio $g(4)/f(n)$? This also has limiting value 1.

In 5.2 you are mixing two objectives: studying distribution of game length, and studying maximum game length. The same empirical study

applies to both, but you should announce the two and then see what your study says about each.

In 5.3, X is not well defined: it will depend on n . The use of probabilistic jargon ($x \in X$, for example) is a bit out of place in this paper. The equation $\dots = 4.93 \dots$ is not correct, of course; what is true is that for certain n you get approximately this value. Same with Var . The graph on p 13 shows the bimodality clearly; I would not have thought this was going to be normally distributed. What is interesting, though, is that apparently most of the paths have length 4 for large n . This might be explainable. I felt that this section was tacked on; I think you guys have enough results without this, other than a report on the discovery of the longest games for certain small n and the appearance that 4 is the mode.

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