Set Theory and Logic: Fundamental Concepts
(Notes by Dr. J. Santos)

A.1. **Primitive Concepts.** In mathematics, the notion of a set is a primitive notion. That is, we admit, as a starting point, the existence of certain objects (which we call sets), which we won't define, but which we assume satisfy some basic properties, which we express as axioms. In other words, we won't describe what a set is, we will describe what can be done with sets.

Intuitivelly, a set is a collection of objects of any kind, which we call the elements of a set. The second primitive notion of set theory is the notion of belonging. We write $x \in X$ meaning 'x belongs to the set X', or 'x is an element of X' (Tipically we use capital letters to designate sets and small letters to designate elements of a set).

The first axiom of set theory is

Axiom 1a. A set is determined by its elements

Remark 1. It is important to notice that this axiom is a non trivial assertion about belonging. To understand this consider an analogous situation in which we consider human beings in the place of sets and elements, and $x \in A$ means x is an ancestor of A. Then clearly A is not determined by its ancestors.

So to describe a set we only need to list its elements. For example, if we have three objects a, b, c, the set whose elements are precisely a, b, c is denoted by $\{a, b, c\}$.

Remark 2. We should point out that the existence of the set $\{a, b, c\}$ is not a given. It is rather a consequence of other axioms of set theory, concerned with the existence of sets. This is not the place, however, to go into those matters so we will just assume that every set we talk about exists.

Other examples of sets are

(1) $\mathbb{Z}$ is the set of integers $0, -1, 1, -2, 2, \ldots$
(2) $\mathbb{Z}_+$ is the set of positive integers $1, 2, 3, 4, \ldots$
(3) $\mathbb{Q}$ is the set of rational numbers $1, \frac{1}{2}, 3, \frac{1}{5}$, etc.
(4) $\mathbb{R}$ is the set of real numbers $\sqrt{2}, \pi, 2, e, \frac{2}{3}$, etc.

A.2. **Designations and Sentences.** $7, 3+4, \{3, 4\}, \mathbb{Z}, \mathbb{R}$ are all examples of what we call designations. Designations are names used to refer to mathematical objects: numbers, points, geometric figures, sets and their elements, etc. Observe that 7 and $3+4$ refer to the same object; we indicate this fact by writing $7 = 3 + 4$.

$7 = 3 + 4,\ 4 \leq 4,\ 2 + 1 < 1 + 2,\ \sqrt{2} \in \mathbb{Z}$ are all examples of what we call sentences. Sentences express statements pertaining to the mathematical objects. These statements can be either true or false. For example the first two sentences are true, and the last two sentences are false.

Exercise 1. Which of the following sentences are true?

$$1 = \{1\},\ 1 \in \{1\},\ \{1\} \in 1,\ \{1\} \in \{1\}$$

'True' and 'false' are called the logical values of a sentence. Two sentences are said to be equivalent if they have the same logical value. For example, the following sentences are equivalent:

$$7 \leq 0,\ (-2)^5 = 2^5$$

To indicate that two sentences - designated by, say, the symbols p and q - are equivalent, we write $p \Leftrightarrow q$. We also say with the same meaning that 'p if and only if q'. Following a common practice, we will often abreviate 'if and only if' to 'iff'.

Notice that $p \Leftrightarrow q$ is itself a sentence: it is true when p and q have the same logical value, and false when they have different values.

Using the notion of equivalence of sentences we may rewrite the first axiom as

Axiom 1b. Consider two sets A and B. Then $A = B$ if and only if $x \in A \Leftrightarrow x \in B$.

A.3. **Operations on sets.** Given two sets U, V, a natural thing to do is to join the elements of U and V into a single set:

Definition 1 (Union of two sets)**.** Let U, V be sets. The union of U and V, $U \cup V$, is the set whose elements x are characterized by

$$x \in U \cup V \Longleftrightarrow (x \in U \text{ or } x \in V)$$

For example $\{1, \mathbb{Z}\} \cup \{\mathbb{Z}, 2\} = \{1, \mathbb{Z}, 2\}$.

We should stop here to clarify the meaning of 'or', which is somehow ambiguous in everyday language. If p and q are sentences then 'p or q' means that at least one of the given sentences p, q is true. Hence, the sentence 'p or q' is only false if both p and q are false.

Other natural way of building a set is to consider the elements common to both sets U and V:

Definition 2 (Intersection of Two Sets)**.** We define the intersection of two sets U, V, and write $U \cap V$, as the set whose elements are characterized by

$$x \in U \cap V \Longleftrightarrow x \in U \text{ and } x \in V$$

For example $\{\mathbb{R}, \{1\}\} \cap \{\{1\}, 2\} = \{\{1\}\}$.

The meaning of 'and' is the usual one in everyday language: If p and q are sentences then 'p and q' means that both p and q are true.

If A and B have no elements in common then $A \cap B$ is a set with no elements:

Definition 3. The empty set, $\emptyset$, is the set containing no elements (hence the sentence $x \in \emptyset$ is always false). By the first axiom there is only one such set. If two sets A, B have no common elements, that is if $A \cap B = \emptyset$, they are called disjoint.

Proposition 1. *Let A be any set. Then*

1. $A \cap \emptyset = \emptyset$*;*
2. $A \cup \emptyset = A$.

Proof.

1. By axiom 1b, $(A \cap \emptyset = \emptyset) \Longleftrightarrow (x \in A \text{ and } x \in \emptyset \Leftrightarrow x \in \emptyset)$. So we want to show that the sentence '$(x \in A \text{ and } x \in \emptyset) \Leftrightarrow x \in \emptyset$' is true.

   Since $x \in \emptyset$ is false, we get the sentence '$x \in A$ and False $\Leftrightarrow$ False' which is a true sentence.
2. Now we want to show that '$(x \in A \text{ or } x \in \emptyset) \Leftrightarrow x \in A$' is a true sentence. We have two cases:
   (a) $x \in A$ is true. Then we get 'True or False $\Leftrightarrow$ True' which is a true sentence.
   (b) $x \in A$ is false. Then we get 'False or False $\Leftrightarrow$ False' which is also a true sentence.

This concludes the proof. □

Exercise 2. Show that

(1) $A \cup A = A$;
(2) $A \cap A = A$.

A.4. **Subsets.**

Definition 4. We say U is a subset of V (and write $U \subset V$) when its elements are also elements of V:

$$U \subset V \iff (\text{if } x \in U \text{ then } x \in V)$$

Again, we should stop to clarify the meaning of the words 'if - then -'. Given two sentences p, q, the sentence 'if p then q' (often denoted by $p \Rightarrow q$) is only false when p is true and q is false. So, for example, among the sentences

$$2 \geq 2 \Rightarrow 3 > 2 + 1,\ 3 = 2 \Rightarrow 5 < 0,\ 3 = 2 \Rightarrow 5 \geq 0$$

only the first is false.

Proposition 2. *Given sets A, B we have*

(1) $\emptyset \subset A$*;*
(2) $A \subset A$*;*
(3) $A \cap B \subset A$*;*
(4) $A \subset A \cup B$*;*
(5) *If $A \subset B$ and $B \subset C$ then $A \subset C$.*

Proof. Proofs of $p \Rightarrow q$ usually proceed as follows: when p is false the sentence is automatically true so we begin by assuming that p is true. From this we derive then that q is also true.

(1) We have to show that '$x \in \emptyset \Rightarrow x \in A$' is true. Since $x \in \emptyset$ is false we are done.
(2) We have to show that '$x \in A \Rightarrow x \in A$'. Assume the left hand side, $x \in A$, is true. We want to show the right hand side, $x \in A$, is also true. This is clearly the case.
(3) We have to show that $(x \in A \text{ and } x \in B) \Rightarrow x \in A$. Assume both $x \in A$ and $x \in B$ are true. We want to show that $x \in A$ is true. That's clearly the case.
(4) We have to show that $x \in A \Rightarrow (x \in A \text{ or } x \in B)$. Assume $x \in A$ is true. We want to show '$x \in A$ or $x \in B$' is true, that is, at least one of them is true. This is the case since $x \in A$ is true.
(5) We want to show that

$$(x \in A \Rightarrow x \in B) \text{ and } (x \in B \Rightarrow x \in C) \Longrightarrow (x \in A \Rightarrow x \in C)$$

So we assume the following two sentences are true
  i. $x \in A \Rightarrow x \in B$
  ii. $x \in B \Rightarrow x \in C$
Now we want to show that $x \in A \Rightarrow x \in C$ so we assume that
  iii. $x \in A$
is true and try to prove that $x \in C$ is true. Now i. and iii. together show that $x \in B$ is true and this together with ii. shows that $x \in C$ is true, as desired.

This finishes the proof □

A.5. **Proving equivalences.** Almost all proofs of equivalence of two sentences p, q are divided into two steps: first to show that $p \Rightarrow q$ and second to show that $q \Rightarrow p$. This procedure uses the following result:

Exercise 3. Show that

$$(p \Rightarrow q) \text{ and } (q \Rightarrow p) \Longleftrightarrow (p \Leftrightarrow q)$$

In the same way, most proofs of equality betwen two sets A, B are done by first showing that $A \subset B$ and then showing that $B \subset A$:

Exercise 4. Show that

$$A = B \Longleftrightarrow (A \subset B \text{ and } B \subset A)$$

As an example of this procedure we will prove the following

Proposition 3. *Given sets X, Y we have*

$$X \subset Y \Longleftrightarrow X \cap Y = X$$

Proof. We divide the proof into two steps:

(1) We begin by showing that $(X \cap Y = X) \Longrightarrow X \subset Y$. So we assume that $X \cap Y = X$ and we want to show that $X \subset Y$.

    We saw in proposition 2 that $X \cap Y \subset Y$. Since $X \cap Y = X$ it follows that $X \subset Y$.

(2) Now we will show that $X \subset Y \Longrightarrow X \cap Y = X$. Assume $X \subset Y$. We want to show that $X \cap Y = X$. We'll do it in two steps:
   (a) $X \cap Y \subset X$ was already proven in proposition 2.
   (b) To show that $X \subset X \cap Y$ let $x \in X$. Then, since $X \subset Y$, it follows that $x \in Y$. Since x is an element of both X and Y, it follows that $x \in X \cap Y$.

This concludes the proof. □

Exercise 5. Show that

(1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
(3) $A \subset B \Longleftrightarrow A \cup B = B$.

A.6. **Negation.** If p is a sentence then the negation of p, 'not p', is the sentence that states that p is false. Thus, 'not p' and p have always different logic values.

We use the designations $a \neq b$ for 'not $a = b$', $x \notin X$ for 'not $x \in X$' and $A \not\subset B$ for 'not $A \subset B$'.

One very useful result is the following:

Exercise 6. Show that

$$(p \Rightarrow q) \Longleftrightarrow (\text{not } q) \Rightarrow (\text{not } p)$$

This formula is quite handy since sometimes it is much easier to prove the right hand side than it is to prove the left hand side. It is thus important to know how to negate sentences. We have the following properties (known as De Morgan laws):

Exercise 7. Show that

(1) not $(p \text{ and } q) \Longleftrightarrow (\text{not } p) \text{ or } (\text{not } q)$
(2) not $(p \text{ or } q) \Longleftrightarrow (\text{not } p) \text{ and } (\text{not } q)$

The corresponding rule for the implication is

Exercise 8. Show that

$$\text{not } (p \Rightarrow q) \Longleftrightarrow p \text{ and } (\text{not } q)$$

Closely associated with negation is the difference of sets:

Definition 5. We define the difference of two sets U, V, and write $U - V$, as the set whose elements are characterized by

$$x \in U - V \Longleftrightarrow x \in U \text{ and } x \notin V$$

To better express the relationship betwen negation and difference of sets we will introduce the notion of universal set. Very often it happens that we can fix a set E such that all the sets we are considering are subsets of E. This set is then called the universal set. It is then often useful to refer to the set of all the subsets of E:

Definition 6. Given a set E, the collection of all the subsets of E forms a set, denoted by $\mathcal{P}(E)$ or 2^E. This set is called the power set of E. We thus have

$$A \in \mathcal{P}(E) \Longleftrightarrow A \subset E$$

Exercise 9. Show that $\mathcal{P}(E) \cap \mathcal{P}(F) = \mathcal{P}(E \cap F)$ and $\mathcal{P}(E) \cup \mathcal{P}(F) \subset \mathcal{P}(E \cup F)$.

In this special case, we use a special notation for the complement: $E - A = A^c$. Then, $x \in A^c \Leftrightarrow x \in E$ and $x \notin A$. In this context, $x \in E$ is always a true sentence, hence $x \in A^c \Leftrightarrow x \notin A$. Then from the rules for the negation of sentences we can derive several identities:

Exercise 10. Show that
(1) $A - B = A \cap B^c$
(2) $(A^c)^c = A$
(3) $A \subset B \Longleftrightarrow B^c \subset A^c$
(4) $\emptyset^c = E$, $E^c = \emptyset$
(5) $A \cap A^c = \emptyset$, $A \cup A^c = E$
(6) $(A \cup B)^c = A^c \cap B^c$
(7) $(A \cap B)^c = A^c \cup B^c$

A.7. **Expressions with Variables.** In mathematical language it is frequent the use of expressions which depend on one or more variables, that is, symbols (usualy letters) which can be substituted by elements of a certain set. For example, consider the expressions

$$x, \ (x - y)^2, \ x^2 - 2xy + y^2$$

where x, y can be subsituted by elements of the set of real numbers $\mathbb{R}$. Now consider the sentences

$$x^2 > 0, \ 2^x = x^2, \ x^2 - y^2 = 0, \ x - y > y - z$$

with $x, y, z \in \mathbb{R}$. These sentences will be true or false depending on the value of the variables x, y, z. In general, given a set X and a sentence $p(x)$, with x taking values in X, three possibilities may occur:
(1) $p(x)$ is true for all $x \in X$
(2) $p(x)$ is false for all $x \in X$
(3) There exist $x, y \in X$ such that $p(x)$ is true and $p(y)$ is false

To distinguish betwen these 3 cases we introduce the following symbols, called quantifiers:

- $\underset{x \in X}{\forall} p(x)$ states that for any $x \in X$, $p(x)$ is true; that is, we are in case (1).
- $\underset{x \in X}{\exists} p(x)$ states that there is at least one element $x \in X$ such that $p(x)$ is true; that is, we are either in case (1) or case (3).

For example, the following sentences are all true:

$$\underset{x \in \mathbb{R}}{\forall} x^2 + 1 > 0, \ \underset{x \in \mathbb{R}}{\exists} x^4 \leq 0, \ \underset{x \in \mathbb{R}}{\exists} x^2 - 3 = 0$$

Now, if we negate $\underset{x \in X}{\forall} p(x)$ then we are not in case (1) so we are either in case (2) or in case (3). That is, $\underset{x \in X}{\exists}$ not $p(x)$. In a similar fashion, it follows that, if we negate $\underset{x \in X}{\exists} p(x)$, then we are not in case (1) and we are not in case (3). Hence we are in case (2), which is $\underset{x \in X}{\forall}$ not $p(x)$. These laws, known as the second De Morgan laws, are of fundamental importance. We write them again together:

$$\text{not}\left(\underset{x \in X}{\forall} p(x)\right) \Longleftrightarrow \underset{x \in X}{\exists} \text{ not } p(x)$$

$$\text{not}\left(\underset{x \in X}{\exists} p(x)\right) \Longleftrightarrow \underset{x \in X}{\forall} \text{ not } p(x)$$

For example,

$$\text{not}\left(\underset{x \in \mathbb{R}}{\forall} x^2 > 0\right) \Longleftrightarrow \underset{x \in \mathbb{R}}{\exists} x^2 \leq 0$$

$$\text{not}\left(\underset{x \in \mathbb{R}}{\forall} \underset{y \in \mathbb{R}}{\forall} \underset{z \in \mathbb{R}}{\exists} x = yz\right) \Longleftrightarrow \underset{x \in \mathbb{R}}{\exists} \underset{y \in \mathbb{R}}{\exists} \underset{z \in \mathbb{R}}{\forall} x \neq yz$$

Exercise 11. Show that

$$A \subset B \Longleftrightarrow \underset{x \in A}{\forall} x \in B$$

Now we look at a more complicated example. Consider the sentence $\underset{x \in \mathbb{R}}{\forall} \underset{y \in \mathbb{R}}{\exists} y \leq x$. It states that, given any number x we can always find a number y such that $y \leq x$. This sentence is clearly true: given any x we can take for example $y = x$. Then clearly $y \leq x$. Consider what happens if we switch the quantifiers: we get the sentence $\underset{y \in \mathbb{R}}{\exists} \underset{x \in \mathbb{R}}{\forall} y \leq x$. This sentence states that there is a number y which is smaller or equal to any other number. This certainly seems false. But to prove it, it is not enough to claim that we cannot find any such number! The easier way to show this sentence is false is to show that its negation is true. The negation of the sentence is $\underset{y \in \mathbb{R}}{\forall} \underset{x \in \mathbb{R}}{\exists} y > x$. This sentence is clearly true: given any y choose for example $x = y - 1$.

We just saw that exchanging the order of the quantifiers $\forall$ and $\exists$ does not produce equivalent sentences. On the other hand, exchanging the order of quantifiers of the same type always gives an equivalent sentence. For example, the propositions

$$\underset{x \in \mathbb{R}}{\forall} \underset{y \in \mathbb{R}}{\forall} (x^3 = y^3 \Leftrightarrow x = y)$$

$$\underset{y \in \mathbb{R}}{\forall} \underset{x \in \mathbb{R}}{\forall} (x^3 = y^3 \Leftrightarrow x = y)$$

are equivalent, and we can also write

$$\underset{x, y \in \mathbb{R}}{\forall} (x^3 = y^3 \Leftrightarrow x = y)$$

Remark 3. It is common to write simply $p(x) \Rightarrow q(x)$ with the meaning that $\underset{x\in X}{\forall}(p(x) \Rightarrow q(x))$. In the same way it is common to write $p(x) \Leftrightarrow q(x)$ instead of $\underset{x\in X}{\forall}(p(x) \Leftrightarrow q(x))$. As examples, we have the true sentences

$$x < 1 \Rightarrow x < 3$$
$$(x < y \text{ and } y < z) \Rightarrow x < z$$
$$x^2 > 0 \Leftrightarrow x \neq 0$$
$$(x > 3 \text{ or } x = 3) \Leftrightarrow x \geq 3$$

Exercise 12. Show that the following are equivalent sentences:
(1) $A \subset B$
(2) For any set C, $(B \cap C) \cup A = B \cap (C \cup A)$
(3) There is a set C such that $(B \cap C) \cup A = B \cap (C \cup A)$

A.8. **Collections of Sets.** As a direct application of quantifiers we can generalize the notions of union and intersection to arbitrary collections of sets (the name 'collection of sets' is a way to refer to a set whose elements are also sets, for example $\mathcal{P}(E)$). Let $\mathcal{C}$ be a collection of sets. Then we can join all the elements of all the sets in $\mathcal{C}$ into a new set:

Definition 7. We define the union $\bigcup_{X\in\mathcal{C}} X$ as the set whose elements are characterized by

$$x \in \bigcup_{X\in\mathcal{C}} X \iff \underset{X\in\mathcal{C}}{\exists}\ x \in X$$

We can also consider the elements common to all sets in $\mathcal{C}$:

Definition 8. Let $\mathcal{C}$ be a nonempty collection of sets. We define the intersection $\bigcap_{X\in\mathcal{C}} X$ as the set whose elements are characterized by

$$x \in \bigcap_{X\in\mathcal{C}} X \iff \underset{X\in\mathcal{C}}{\forall}\ x \in X$$

Exercise 13. Let $\mathcal{C} = \{A, B\}$. Show that

$$\bigcup_{X\in\mathcal{C}} X = A \cup B, \quad \bigcap_{X\in\mathcal{C}} X = A \cap B$$

Exercise 14. Show that $\bigcup_{X\in\emptyset} X = \emptyset$

A.9. **Sets defined by sentences.** Given a set X, a very important way to build subsets of X is the following: consider a sentence $p(x)$ where x takes values on the set X. Then we build the subset of X whose elements are exactly those for which $p(x)$ is true. We denote this set by $\{x \in X \,|\, p(x)\}$:

$$y \in \{x \in X \,|\, p(x)\} \iff (y \in X \text{ and } p(y))$$

One of the more important axioms of set theory is the following:

Axiom 2. The set $\{x \in X \,|\, p(x)\}$ exists.

Exercise 15. Show that
(1) $A \cap B = \{x \in A \,|\, x \in B\}$;
(2) $A - B = \{x \in A \,|\, x \notin B\}$.

Exercise 16. Show that

$$\{x \in X \,|\, p(x)\} \subset \{x \in X \,|\, q(x)\} \Longleftrightarrow \underset{x \in X}{\forall}\; p(x) \Rightarrow q(x)$$

Exercise 17. Given a collection of sets $\mathcal{C} \neq \emptyset$ show that, for any $A \in \mathcal{C}$,

$$\bigcap_{X \in \mathcal{C}} X = \left\{ x \in A \,|\, \underset{X \in \mathcal{C}}{\forall}\; x \in X \right\}$$

We will now prove that the intersection of an empty collection of sets is not a set (the only example of a non-existing set we will encounter):

Theorem (Russell's paradox)**.** $\bigcap_{X \in \emptyset} X$ *is not a set.*

Proof. We will prove by contradiction. This method consists in assuming the result we want to prove is false and arriving at a contradiction. The contradiction shows that our assumption was wrong, hence the result is true. So assume $A = \bigcap_{X \in \emptyset} X$ is a set. By axiom 2 we can build the set

$$B = \{Y \in A \,|\, Y \notin Y\}$$

Then, by definition of B, the following is a true sentence:

$$B \in B \Longleftrightarrow (B \in A \text{ and } B \notin B)$$

It follows that both sentences $B \notin A$ and $B \notin B$ must be true. Then, by definition of A,

$$B \notin A \Leftrightarrow \text{not} \left(\underset{X \in \emptyset}{\forall}\; B \in X \right) \Leftrightarrow \underset{X \in \emptyset}{\exists}\; B \notin X$$

But this last sentence is false so we have reached a contradiction. We conclude A is not a set. $\square$

This problem disappears if we restrict our attention to subsets of a fixed universal set E. Then, we define, for an arbitrary collection of sets $\mathcal{C} \subset \mathcal{P}(E)$,

$$\bigcap_{X \in \mathcal{C}} X = \left\{ x \in E : \underset{X \in \mathcal{C}}{\forall}\; x \in X \right\}$$

Exercise 18. Show that, with this new definition, we have

$$\bigcap_{X \in \emptyset} X = E$$

References

These notes were, in several places, inspired by the presentation of the material in the following sources:

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