The Hurewicz Theorem

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1 Introduction

The fundamental group and homology groups both give extremely useful information, particularly about path-connected spaces. Both can be considered as functors, so we can use these constructional invariants as convenient guides to classifying spaces. However, though homology groups are often easy to compute, the fundamental group sometimes is not. In fact, it is often not even obvious when a space is simply connected. In particular, noncontractible simply connected spaces are difficult to identify, as contractibility is often the most geometrically intuitive way to determine if a space is simply connected.

As such, we would like to know if there is a connection between these two seemingly disjoint geometric concepts. The answer lies in the Hurewicz theorem, which in general gives us a connection between generalizations of the fundamental group (called homotopy groups) and the homology groups. As we will show, there exists a "Hurewicz homomorphism" from the nth homotopy group into the nth homology group for each n, and the Hurewicz theorem gives us information about this homomorphism for specific values of n. For the particular case of the fundamental group, the Hurewicz theorem indicates that the Hurewicz homomorphism induces an isomorphism between a quotient of the fundamental group and the first homology group, which provides us with a lot of information about the fundamental group. In many cases, the Hurewicz theorem tells us that the Hurewicz homomorphism is actually an isomorphism between the lowest nontrivial homotopy group and the lowest nontrivial reduced homology group. Moreover, it gives us a method for determining which homotopy group is the lowest nontrivial one given the homology groups. This connection is a powerful computational tool, despite the fact that it tells us little about homotopy groups past the lowest nontrivial one.

In this paper, we first develop and prove a special case of the Hurewicz theorem. We then give a few results from the theory of the higher homotopy groups. Finally, we state the full form of the Hurewicz theorem (without proof). We discuss some applications throughout the paper.

2 The Fundamental Group and First Homology Group

The simplest case of the Hurewicz theorem, which in general relates the nth homotopy group (to be defined later for $n \neq 1$) and the nth homology group, is the $n = 1$ case. We develop this, state the Hurewicz theorem for this case, and give an application. We then prove this case, which is not too difficult thanks to the lucky coincidence that paths and singular 1-simplices are essentially the same objects.

2.1 Statement and Application of the $n = 1$ Hurewicz Theorem

We need very little machinery to prove the $n = 1$ case of the Hurewicz theorem, but we do need two simple, related group theoretical notions from [1].

Definition 2.1. *The* ***commutator subgroup*** *of a group G, denoted $[G, G]$, is the subgroup generated by all elements, called* ***commutators****, of the form $ab(ba)^{-1}$ for any a and b in G.*

Commutators are, in a sense, a measure how much of G fails to be commutative. In particular, the commutator subgroup is trivial if and only if all commutators are 1. But a commutator is trivial if and only if $(ba)^{-1}$ is equal to $(ab)^{-1}$. By the uniqueness of inverses, $(ba)^{-1} = (ab)^{-1}$ implies $ba = ab$. Hence all commutators are trivial, that is, the commutator subgroup is trivial, if and only if G is abelian.

Notice that the commutator subgroup is normal by the following chain of equalities for any $g \in G$:

$$\begin{aligned} g^{-1}(aba^{-1}b^{-1})g &= g^{-1}agg^{-1}bgg^{-1}a^{-1}gg^{-1}b^{-1}g \\ &= (g^{-1}ag)(g^{-1}bg)(g^{-1}a^{-1}g)(g^{-1}b^{-1}g) \\ &= (g^{-1}ag)(g^{-1}bg)(g^{-1}ag)^{-1}(g^{-1}bg)^{-1} \end{aligned}$$

The last line is in $[G, G]$, so conjugating a generator of $[G, G]$ keeps us in $[G, G]$. Breaking a general element into a product of generators and inserting copies of the identity element between them means that $[G, G]$ is closed under conjugation of any element, which establishes normality. Hence, we may quotient out:

Definition 2.2. *The* ***abelianization*** *G_{ab} of a group G is the quotient $G/[G, G]$.*

Unsurprisingly, the abelianization is appropriately named.

Proposition 2.3. *The abelianization G_{ab} of a group G is abelian.*

Proof. For any a, b in G, we have that $aba^{-1}b^{-1}$ is in the commutator subgroup by definition. Hence the class of $aba^{-1}b^{-1}$ under the quotient map is the class of the identity 1, so right multiplying by the class of ba shows that the class of ab equals the class of ba.

We said before that the commutator subgroup of an abelian group is trivial. This means that the abelianization of an abelian group G is G itself. It also happens that abelianization is also a functor.

Note that any homomorphism $h : \pi_1(X, x_0) \to G$, where G is any abelian group, induces a homomorphism $h' : (\pi_1)_{ab}(X, x_0) \to G$, specifically because G is abelian.

We now construct a homomorphism $h : \pi_1(X, x_0) \to H_1(X)$ which will be used in the statement of the Hurewicz theorem. Let X be a topological space base pointed at x_0. Pick an element $[\gamma]$ of $\pi_1(X, x_0)$, and let γ be a path in this class. This map γ goes from I to X, where I is the 1-simplex, so γ can be thought of as a singular 1-simplex in X. But $\gamma(0) = \gamma(1) = x_0$, where 0 and 1 are the two faces of the 1-simplex I. Hence the boundary of γ is 0, so γ is a cycle. We then define $h([\gamma])$ to be the homology class of γ (considered as a cycle).

We can now state the $n = 1$ case of the Hurewicz theorem, Theorem 2A.1 in [2] and Theorem 7.1 in [3]:

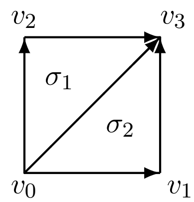

Figure 1: Splitting of I to show h is well-defined.

Theorem 2.4. *(Hurewicz, $n = 1$ case): If X is path connected and base pointed at x_0, then h is a well-defined homomorphism, and the map $h' : (\pi_1)_{ab}(X, x_0) \to H_1(X)$ induced by h is an isomorphism.*

Before we prove this, we state an obvious corollary and application of the corollary:

Corollary 2.5. *If X is a path connected space with abelian fundamental group, then $\pi_1(X, x_0) = (\pi_1)_{ab}(X, x_0)$ is isomorphic to $H_1(X)$.*

Corollary 2.6. *Let G be a path connected topological group with identity element 1. Then $\pi_1(G, 1)$ is isomorphic to $H_1(G)$.*

The second corollary follows because the fundamental group of a topological group is abelian.

The Hurewicz theorem is trivially false when X is not path connected; take X to be $S^1 \coprod S^1$. Then the fundamental group at any point is $\mathbb{Z}$, but the first homology group is $\mathbb{Z} \bigoplus \mathbb{Z}$.

2.2 Proof of the $n = 1$ Hurewicz Theorem

Proof of Hurewicz theorem, $n = 1$. We follow [2]. We must check the following:

- h is well-defined
- h is a homomorphism
- h' is an isomorphism

We first show that h is well-defined, i.e., that it gives the same homology class regardless of choice of γ. Let γ' be some other path in the homotopy class $[\gamma]$. and, let $H(t, s)$ be a homotopy from γ' to γ. Let the vertices of $I \times I$ be v_i for $i = 0, 1, 2, 3$, ordered as in Figure 2.2, and split $I \times I$ on the v_0, v_3 diagonal with all sides oriented away from v_0 and towards v_3. Then H can be regarded as the sum of two singular 2-simplices σ_1 and σ_2, where σ_1 is on the v_2 side. (Again see Figure 2.2.) We take the boundaries, and let f_{x_0} be the constant 1-simplex which maps to x_0:

$$\begin{aligned} \partial\sigma_1 &= f_{x_0} - D + \gamma' \\ \partial\sigma_2 &= \gamma - D + f_{x_0} \end{aligned}$$

where D is the restriction of H to the diagonal. Subtracting gives $\partial(\sigma_1 - \sigma_2) = \gamma' - \gamma - 2f_{x_0}$. As the constant map from Δ^1 to x_0 is the boundary of the constant map from Δ^2 to x_0, we conclude that $\gamma' - \gamma$ is a boundary. Adding this to γ gives us γ', so γ' is equal to γ plus a boundary. Hence the homology class of γ' is the homology class of γ.

Next, we check that h is a homomorphism. To this end, let $[\gamma]$ and $[\gamma']$ be elements of the fundamental group. Write Δ^2 as $[v_0, v_1, v_2]$, and let $\sigma : \Delta^2 \to X$ be the map which first takes the orthogonal projection of Δ^2 onto the $[v_0, v_2]$ edge and then applies $\gamma\gamma' : I = \Delta^1 \to X$. Notice that the boundary of σ is $\gamma' - \gamma\gamma' + \gamma$ up to a reparametrization of $\gamma\gamma'$ (which does not affect homotopy). Hence, $h([\gamma]) + h([\gamma']) - \partial\sigma = \gamma\gamma' = h([\gamma][\gamma'])$, which shows that h is a homomorphism.

We note that the homology class of $\bar{\gamma}$ is the homology class of $-\gamma$, where γ is any path, because h is a homomorphism.

To show that h' is an isomorphism, it suffices to show that h is surjective and has kernel equal to the commutator subgroup of $\pi_1(X, x_0)$.

Surjectivity: Let $\sigma = \sum_i n_i\sigma_i$ be a 1-cycle. We can split up multiples of the σ_i so that each n_i is ± 1, though this of course allows us to have $\sigma_i = \sigma_j$ for $i \neq j$. Moreover, by possibly reversing some of the σ_i, we can assume that all the n_i are 1. Thus, without loss of generality we may take $\sigma = \sum_i \sigma_i$.

If there is some σ_i which is not a loop, there must exist a σ_j in the sum such that the composition $\sigma_i\sigma_j$ is defined, or else the boundary of σ would be nontrivial. But $\sigma_i\sigma_j$ is homologous to $\sigma_i + \sigma_j$ as we showed when checking that h is a homomorphism. Therefore we can replace $\sigma_i + \sigma_j$ with the single element $\sigma_i\sigma_j$ without changing the homology class of σ. Without loss of generality then, we may assume that all of the σ_i are loops.

Now let γ_i be any path from the base point x_0 to the base point of σ_i (regarded as a loop), which exists because X is path connected. Then $\gamma_i\sigma_i\bar{\gamma}_i$ is homologous to $\gamma_i + \sigma_i + \bar{\gamma}_i$, since h is a homomoprhism, which is homologous to $\gamma_i + \sigma_i - \gamma_i = \sigma_i$ since $\bar{\gamma}_i$ is homologous to $-\gamma_i$. Thus, by replacing σ with a homologous cycle, we may assume that all the σ_i are loops based at x_0. Finally, since the sum is homologous to the composition, we can take $\sum_i \sigma_i$ to be a single singular 1-simplex without changing the homology class. In particular, this singular 1-simplex is a loop at x_0, which means that its homotopy class maps to the homology class of σ, which is what we needed.

Kernel: First let $\gamma\gamma'\bar{\gamma}\bar{\gamma'}$ be in the commutator subgroup of π_1. Then its image under h is $\gamma + \gamma' + \bar{\gamma} + \bar{\gamma'}$. But $H_1(X)$ is abelian, so this sum is zero (since $\bar{\gamma} = -\gamma$). Hence the commutator subgroup of π_1 is contained in the kernel of h.

Now suppose that $[\gamma]$ is in the kernel of h. It suffices to show that $[\gamma]$ is trivial in $(\pi_1)_{ab}(X, x_0)$. As a loop, γ is a 1-cycle, and it is homologous to 0, which means that γ is the boundary of some 2-cycle $\sigma = \sum_i n_i\sigma_i$. We can, as before, take $n_i = \pm 1$.

Now, for each σ_i, we can write $\partial\sigma_i = \tau_{i0} - \tau_{i1} + \tau_{i2}$ for three 1-cycles τ_{ij}. Notice that

$$\gamma = \partial \sum_i n_i\sigma_i = \sum_i n_i\partial\sigma_i = \sum_i n_i(\tau_{i0} - \tau_{i1} + \tau_{i2}) = \sum_{i,j} (-1)^j n_i\tau_{ij}$$

But γ is a singular 1-cycle, which means that all the τ_{ij} must form canceling pairs except for one, which is equal to γ. If we then glue together the 2-simplices, identifying canceling pairs of edges(preserving orientation), we get a Δ-complex K.

Now since identified pairs are the same map, the σ_i together form a map $\sigma : K \to X$. Let A be the 0-skeleton of K union the segment corresponding to γ. We can slide the image of each vertex along a path from its original image to x_0. This defines a homotopy of A with a new 0-skeleton which maps each point to x_0 and leaves γ unchanged. By the homotopy extension property, since the 0-skeleton plus the segment is a subcomplex of X, we can extend this homotopy to a homotopy defined on all of K. If we now break the deformed K down into its simplices, we get a new chain $\sum_i m_i\sigma_i'$, but every singular 1-simplex τ_{ij}' on the boundary is now a loop at x_0 (because we moved the image of each vertex to x_0).

Now, since $(\pi_1)_{ab}(X, x_0)$ is abelian, we can write $[\gamma] = \sum_{i,j}(-1)^j m_i[\tau_{ij}']$ in the abelianization. We wrote $\partial\sigma_i = \tau_{i0} - \tau_{i1} + \tau_{i2}$, but because h' is a homomorphism, the sums on j can be condensed to give $[\gamma] = \sum_i m_i[\partial\sigma_i]$. For each σ_i, we can deform the image of σ_i to the image of one vertex by sliding the image of one edge through the image of the interior (pulling the other two vertexes along with it). This deformation is a homotopy between $\partial\sigma_i$ and the constant map. Hence $[\partial\sigma_i] = 0$, so $[f] = 0$ in the abelianization of π_1.

Notice in our statement of the Hurewicz theorem that we took X to be path-connected. We only used the fact that X was path connected in proving that h is surjective, so h is still well-defined for arbitrary spaces. However, if X is not path-connected, then a theorem from basic homology theory tells us that $H_1(X)$ is the direct sum of the first homology groups of the path components of X. The elements of $\pi_1(X, x_0)$ are classes of paths inside the path component of x_0, so the image of h is contained in the homology group of this same path component. This shows us that, although we can still define h in general, it does not give us any new information about X beyond the information gained when only considering the path-connected case.

3 The Higher Homotopy Groups

The higher homotopy groups are, like the fundamental group, homotopy classes of maps into a space. Rather than homotopy classes of paths, though, the maps are higher-dimensional analogs. There are two equivalent definitions. The first uses maps from I^n into the space. This is the most natural way to generalize the definition of the fundamental group, and in particular it makes defining the group operation simple. However, it requires a condition on the entire boundary of I^n, making it a little less natural to relate to homology groups. The other definition uses maps from S^n into the space. This version is a bit harder to work with because the group operation becomes more complicated, but the restriction required is only on a single point.

There is also a notion of relative homotopy groups, which allow us to formulate an even more general version of the Hurewicz theorem. The general form does reduce to a Hurewicz theorem for absolute homotopy, however. It may seem odd that we did not consider relative homotopy for the fundamental group, but this is because the group structure for relative homotopy breaks down for the fundamental group. In the following subsections, we develop the theory of the higher absolute and relative homotopy groups. All the development follows Chapter 4 in [2].

3.1 Absolute Homotopy Groups

Definition 3.1. *The* ***nth (absolute) homotopy group*** *of a topological space X base pointed at some point x_0, denoted $\pi_n(X, x_0)$, is a set of equivalence classes of maps from I^n into X, where I is the unit interval, which map ∂I^n to x_0. The equivalence relation is homotopy by a homotopy f_t for which $f_t(\partial I_n) = x_0$ for all t.*

If we define I^0 to be a point with empty boundary, we can extend this definition to $n = 0$. We immediately notice, then, that $\pi_0(X, x_0)$ is simply the set of path components of X (and is independent of base point), as the boundary condition gives no restriction and a map from I^0 to a point x in X is homotopic to the map from I^0 to any other point x' in the path component of x (a path from x to x' gives us a homotopy).

For $n \geq 2$, we will define a group operation $+$ (the notation is used because the operation will make $\pi_n(X, x_0)$ abelian) as follows. Given two maps f and g from I^n to X, define

$$(f+g)(s_1, s_2, ..., s_n) = \begin{cases} f(2s_1, s_2, ..., s_n) & s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, s_2, ..., s_n) & s_1 \in [\frac{1}{2}, 1] \end{cases}$$

If we have a homotopy f_t between f and some other map f', then we can define a homotopy $(f+g)_t$ from $f+g$ to $f'+g$ by replacing $(f+g)$ by $(f+g)_t$ on the left hand side of the above definition and f by f_t on the right hand side. This homotopy means that $+$ is well-defined as an operation on homotopy classes. It is easy to check that $+$ turns $\pi_n(X, x_0)$ into a group. It is important to remember that this is only a definition for $n \geq 2$; there is not a natural way to define a group structure on $\pi_0(X)$. The definition is of course the same for $n = 1$, but the notation is different because the fundamental group need not be abelian.

The only problem with this definition is the fact that the maps $I^n \to X$ and the homotopies have a cumbersome restriction of requiring their value on the entire boundary of I^n to be x_0. There is an alternate way to view the π_n, though it makes the addition operation more cumbersome.

Proposition 3.2. *The homotopy group $\pi_n(X, x_0)$ is naturally isomorphic to the set of maps $S^n \to X$ which, for some fixed q in S^n, map s_0 to x_0.*

Proof. Maps from I^n to X which map ∂I^n to x_0 can be viewed as maps from $I^n/\partial I^n$ to X which map the image of ∂I^n under the quotient map to x_0. But the image under the quotient map of ∂I^n is a point q, and $I^n/\partial I^n$ is S^n.

We now consider several properties of the higher homotopy groups which will be useful to us.

Proposition 3.3. *If X is a path-connected space, and x and y are two points in X, then $\pi_n(X, x)$ is isomorphic to $\pi_n(X, y)$ for all n.*

The isomorphism is not canonical; it will depend on a path from x to y, but each homotopy class of path from x to y will give a unique isomorphism.

Proof. We already know this for the $n = 1$ case, and since there is only one path component of X, the $n = 0$ case is trivial. Now consider any map f in $\pi_n(X, y)$, considered as a map from I^n. Let γ be any path from x to y, and let $S = [\epsilon, 1-\epsilon]^n$ for some ϵ in $(0, \frac{1}{2})$. From each point on ∂I^n, we draw a line segment to $(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$, which intersects ∂S. In particular, these segments are all radial and centered at $(\frac{1}{2}, ..., \frac{1}{2})$. Now define f' by reparametrizing f such that its domain is S; notice that on ∂S, our new map f' must be identically y. We can extend f' to a map $f_\gamma : I^n \to X$ by letting f_γ equal γ on each radial segment (reparametrized appropriately).

We now create the change of basepoint map $b_\gamma : \pi_n(X, y) \to \pi_n(X, x)$ by sending $[f]$ to $[f_\gamma]$. This is well-defined because we can reparametrize homotopies between two maps on I^n such that they become homotopies on S. We have to check that we have an isomorphism.

First, let f and g be any two maps in $\pi_n(X, y)$. Define the map $f+0$ to be f reparametrized to have domain on the $x_1 \leq \frac{1}{2}$ half of S and be equal to y on the other half of S. Define $0+g$ the same way but to be g on the $x_1 \geq \frac{1}{2}$ half of S. Since these are reparametrizations (up to adding the constant map to half the domain, which does not change homotopy class), they are homotopic to f and g respectively.

We now claim that the following is a homotopy of $(f+g)_\gamma$ with $(f+0)_\gamma + (0+g)_\gamma$, which is itself homotopic to $f_\gamma + g_\gamma$:

$$h_t(s_1, ..., s_n) = \begin{cases} (f+0)_\gamma((2-t)s_1, s_2, ..., s_n & s_1 \in [0, \frac{1}{2}] \\ (0+g)_\gamma((2-t)s_1 + t - 1, s_2, ..., s_n) & s_1 \in [\frac{1}{2}, 1] \end{cases}$$

Clearly this map begins at $(f+g)_\gamma$ and ends at $f_\gamma + g_\gamma$ by the formulas. Consider points with $s_1 = \frac{1}{2}$. We need to check that our formulas match. Our two formulas at this point are $(f+0)_\gamma(1 - \frac{t}{2}, s_2, ..., s_n)$ and $(0+g)_\gamma(\frac{t}{2}), s_2, ..., s_n)$. Now, if $\frac{t}{2} < \epsilon$, then both of these become the same point on γ by symmetry. If $\frac{t}{2} > \epsilon$, then both of these points are y because we have entered the relevant halves of S. Therefore, this is continuous and a homotopy, showing that b_γ is a homomorphism.

Now notice that $(\gamma\eta)f$ is homotopic to $\gamma(\eta f)$ by simple reparametrization. Furthermore, if c is the constant path, then cf is homotopic to f, again by reparametrization. Therefore, we have an isomorphism because the inverse map $b_{\bar{\gamma}}$ is well-defined and composes with b_γ to give the identity.

Basepoint independence is useful because homology groups are basepoint independent objects; were the higher homotopy groups basepoint dependent, we would not expect to get a useful relationship with homology groups.

On a somewhat different point, given that the $n = 1$ case of the Hurewicz homomorphism involved the abelianization of the fundamental group, we might expect the abelianizations of higher homotopy groups to appear in a higher-dimensional Hurewicz theorem. However:

Proposition 3.4. *If X is any topological space, $\pi_n(X, x_0)$ is abelian for $n \geq 2$ where x_0 is any base point.*

Proof. $f+g$ is a map which is a reparametrization of f on the $s_1 \leq \frac{1}{2}$ portion of I^n and a reparametrization of g on the $s_1 \geq \frac{1}{2}$ portion. Reparametrize each again to some smaller cube of side length less than $\frac{1}{2}$, and map the rest of the domain to x_0 (which preserves continuity). Then we can, again by reparametrization, translate these domain cubes around each other by moving their center in the $s_3 = \frac{1}{2}$ hyperplane for $n > 2$ and in I^2 itself for $n = 2$; in particular, we can translate the centers of the domain cubes simultaneously along linear paths which move the centers to $s_2 = \frac{1}{4}$ and $\frac{3}{4}$, then over to $s_1 = \frac{3}{4}$ and $\frac{1}{4}$, then to $s_2 = \frac{1}{2}$. This exchanges the locations of the domain cubes, and since their side lengths are less than $\frac{1}{2}$, they never intersect in the process of their "motion," so we always have a well-defined continuous map. We can then reparametrize once more to expand the domains back to their original sizes, but they have now switched sides, so the new path is $g+f$. See Figure 2 for a schematic of the process.

Since reparametrization (and adding the constant map at the base point to part of the domain) does not affect homotopy class, we conclude that $f+g$ is homotopic to $g+f$, so that $\pi_n(X, x_0)$ is abelian.

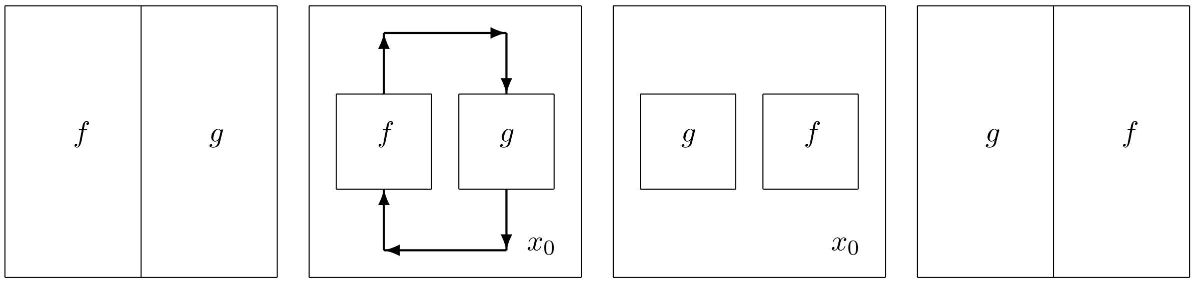

Figure 2: Schematic showing the homotopy making π_2 abelian. The large box is I^2, and the arrows in the second picture are the "routes" taken by the regions on which the function families are f and g. So long as the small boxes are small enough (side length less than $\frac{1}{2}$, they will not intersect each other. Adapted from [2].

The $n = 1$ version of the Hurewicz theorem was slightly inconvenient because computing the first homology group was not sufficient to find the entire fundamental group; we only got a quotient of it. We should expect this inconvenience to disappear in higher-dimensional cases because the quotient (the abelianization) is trivial.

3.2 Relative Homotopy

Another useful notion is that of relative homotopy. Relative homotopy groups allow us to generalize the Hurewicz theorem to relative homology, despite the fact that, as we will see, this does not work as well for $n < 3$. We will not define relative homotopy for $n = 0$; a definition is not immediately obvious, nor is it particularly useful. To make our definition, suppose that $A \subset X$ and the base point x_0 under consideration is in A. We use the cubical domain definition of the homotopy groups. We consider I^{n-1} to be the $s_n = 0$ face of I^n, and define J^{n-1} to be the closure of $\partial I^n - I^{n-1}$. Notice that J^{n-1} is the union of all of the other faces of I^n including the relevant edges and vertices.

Definition 3.5. *The **nth relative homotopy group** $\pi_n(X, A, x_0)$ is the set of homotopy classes of maps $I^n \to X$ such that the maps and homotopies all map I^{n-1} into A and map J^{n-1} to x_0.*

We need to invent a well-defined operation on these sets to turn them into groups. For $n \geq 2$, we can simply restrict the group operation on $\pi_n(X, x_0)$, as s_n does not play a part in the group operation. Consider, however, the $n = 1$ case. I^0 is the point 0 and J^0 is the point 1, so $\pi_1(X, A, x_0)$ is the set of homotopy classes of paths from any point in A to x_0. Just as arbitrary paths from a basepoint do not naturally form a group structure, neither does $\pi_1(X, A, x_0)$.

We can also apply the exact same argument that we used for the absolute homotopy groups to show that relative homotopy groups are abelian, but only for $n > 2$. For $n = 2$, one edge of each domain has a broader restriction on where it maps, so our argument breaks down, as we need to translate the domain along the s_2 direction. The $n = 2$ case is not generally abelian.

Just as in the case of absolute homotopy groups, we can also define the relative homotopy groups in terms of maps from spheres. Maps with classes in the relative homotopy group effectively collapse J^{n-1} to a point, which turns the face I^{n-1} into S^{n-1} and I^n into D^n. Hence we can think of $\pi_n(X, A, x_0)$ as homotopy classes of maps from D^n to X which send $\partial D^n = S^{n-1}$ into A and send a fixed point on ∂D^n to x_0.

We now can extend the idea of path-connectedness and simply-connectedness using the higher homotopy groups.

Definition 3.6. *A space X is* ***n-connected*** *if $\pi_i(X, x_0) = 0$ for $i \leq n$. A pair (X, A) is* ***n-connected*** *if $\pi_i(X, A, x_0) = 0$ for all $i \leq n$.*

Notice that for absolute homotopy, 0-connected precisely means path-connected, whereas 1-connected precisely means simply-connected.

4 The General Hurewicz Homomorphism

Now that we have devloped the formalism of the higher homotopy groups and verified some useful properties, we state the full form of the Hurewicz theorem, which is Theorem 4.37 in [2]. We will not prove it, as the proof is very involved. However, we will construct the Hurewicz homomorphism.

To do the construction, we use the sphere definition of the higher homotopy groups. Let $A \subset X$ be arbitrary topological spaces. Let α be any generator of $H_i(D^i, \partial D^i)$, a group which is isomorphic to $\mathbb{Z}$, and let $[f]$ be an element of $\pi_i(X, A, x_0)$ represented by the map f. Then f induces a map $f_* : H_i(D^i, \partial D^i) \to H_i(X, A)$. We define $h : \pi_i(X, A, x_0) \to H_i(X, A)$ by sending $[f]$ to $f_*(\alpha)$. This is well-defined because homotopic maps induce the same map on homology.

Theorem 4.1. *(Hurewicz) For all $i \geq 1$, the map h is a homomorphism. If $A \subset X$ is an $(n-1)$-connected pair of path-connected spaces with $n \geq 2$ and A nonempty, then the map induced by h on the abelianization $(\pi_n)_{ab}(X, A, x_0)$ into $H_n(X, A)$ is an isomorphism. Furthermore, $H_i(X, A) = 0$ for $1 \leq i < n$.*

We need to assume $n \geq 2$ so that $\pi_n(X, A, x_0)$ is a group. Moreover, h itself is an isomorphism when $n \geq 3$ because these relative homotopy groups are abelian. It is also interesting to notice that we do get a relationship between the ith homotopy group and the ith homology group in general, though it is not always an isomorphism.

With these points in mind, we examine a few useful corollaries. From this point on, we use the term "Hurewicz homomorphism" to mean, for $n \geq 2$, the map $h : \pi_i(X, x_0) \to H_i(X)$ which results from taking A to be a point. For $n = 1$, we mean the isomorphism between $(\pi_1)_{ab}(X, x_0)$ and $H_1(X)$ which we discussed in Theorem 2.4.

Corollary 4.2. *If X is $(n-1)$-connected, then the Hurewicz homomorphism is an isomorphism. Furthermore, $\tilde{H}_i(X) = 0$ for $i < n$.*

We claim that taking A to be the point x_0 reduces us to absolute homotopy groups, which implies the corollary. This is true because the relative homotopy group is the set of homotopy classes of maps which send ∂I^n and $J^{n-1} \subset \partial I^n$ to x_0, which is exactly the absolute homotopy group. Also, the Hurewicz homomorphism is an isomorphism in the absolute case because $\pi_n(X, x_0)$ is abelian for $n \geq 2$.

Corollary 4.3. *If X is simply connected, and $\tilde{H}_i(X) = 0$ for $i < n$, then $\pi_i(X, x_0)$ is trivial for $i < n$.*

Corollary 4.2 is a nice generalization of the relationship between the fundamental group and first homology group. In particular, it tells us that the first non-trivial homotopy group gives us the first non-zero homology group in the simply connected case. Moreover, by Corollary 4.3, if we know that X is simply-connected and are able to compute its homology groups, we immediately find the initial trivial homotopy groups. As a simple application, we have another corollary, given that the homology groups of the n-sphere are well-known (and that the n-sphere for $n > 1$ is simply connected):

Corollary 4.4. *The homotopy groups of the n-sphere, $\pi_i(S^n, x_0)$, are trivial for $i < n$, and $\pi_n(X, x_0) = \mathbb{Z}$.*

Unfortunately, the Hurewicz homomorphism does not give us any useful information about the higher homotopy groups. For spheres, for example, the higher homology groups are all 0, whereas the higher homotopy groups are usually nonzero and are often somewhat peculiar groups (e.g. $\pi_7(S^4) = \mathbb{Z} \times \mathbb{Z}_{12}$).

Finally, we examine another application of Hurewicz's theorem which gives a convenient way to calculate the second homotopy group of a base pointed space which has a universal cover. We start with a short lemma.

Lemma 4.5. *Let $(\tilde{X}, \tilde{x}_0)$ be a covering space of an arbitrary space (X, x_0) with covering map p. Then the induced map $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism for $n \geq 2$.*

Proof. Surjectivity comes first. For $n \geq 2$, the sphere S^n is simply connected. This means that for any map $\phi : (S^n, q) \to (X, x_0)$, the induced map on the fundamental group $\phi_* : \pi_1(S^n, q) \to \pi_1(X, x_0)$ satisfies the lifting criterion $\phi_*(\pi_1(S^n, q)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Therefore, ϕ has a lift to the covering space. The class of this lift obviously maps to $[\phi]$ under p_*, giving surjectivity.

Injectivity is as follows. An element of the kernel of p_* is represented by a map $\tilde{f} : (S^n, q) \to \tilde{X}, \tilde{x}_0$ and a homotopy H of $p\tilde{f}$ to the constant loop. We can then lift H to a homotopy from $\tilde{f}$ to the lift of the constant loop, which is the constant loop in the covering space. Hence the kernel of p_* is trivial, giving injectivity.

Proposition 4.6. *Suppose that (X, x_0) is a space with a universal cover $(\tilde{X}, \tilde{x}_0)$ with covering map p. Then there is a canonical isomorphism between $\pi_2(X, x_0)$ and $H_2(\tilde{X})$.*

Proof. By the lemma, p_* is an isomorphism of $\pi_2(X, x_0)$ with $\pi_2(\tilde{X}, \tilde{x}_0)$. But $\tilde{X}$, being the universal cover of X, is simply connected. Hence the Hurewicz theorem applies for $n = 2$, so the Hurewicz homomorphism h is an isomorphism of $\pi_2(\tilde{X}, \tilde{x}_0)$ with $H_2(\tilde{X})$. Therefore hp_* is the isomorphism we want.

References

[1] Wolfram mathworld.

[2] A. Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[3] W.S. Massey. *A Basic Course In Algebraic Topology.* Springer-Verlag, 1991.

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