18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Tensor Products and Algebras (Lecture 11)

Recall that if X is a topological space, then the cohomology $\mathrm{H}^*(X)$ has the structure of an unstable module over the Steenrod algebra $\mathcal{A}$. Moreover, $\mathrm{H}^*(X)$ is equipped with a multiplication which satisfies the Cartan formula:

$$\mathrm{Sq}^n(xy) = \sum_{n=n'+n''} \mathrm{Sq}^{n'}(x)\,\mathrm{Sq}^{n''}(y).$$

In other words, the multiplication map

$$\mathrm{H}^*(X) \otimes \mathrm{H}^*(X) \to \mathrm{H}^*(X)$$

is compatible with the Steenrod operations Sq^n, if we let Sq^n act by the formula

$$\mathrm{Sq}^n(x \otimes y) = \sum_{n=n'+n''} \mathrm{Sq}^{n'}(x) \otimes \mathrm{Sq}^{n''}(y).$$

Our goal in this lecture is to prove that the preceding formula endows $\mathrm{H}^*(X) \otimes \mathrm{H}^*$ with the structure of an unstable module over the Steenrod algebra. Moreover, a similar result is true for any pair M, N of unstable modules over the big Steenrod algebra $\mathcal{A}^{\mathrm{Big}}$.

Definition 1. We let $\mathcal{A}^{\mathrm{Big}}$ denote the big Steenrod algebra, and $\mathcal{U}^{\mathrm{Big}}$ the category of (graded) unstable $\mathcal{A}^{\mathrm{Big}}$-modules.

Let R denote the free $\mathbf{F}_2$-algebra $\mathbf{F}_2[\ldots, \mathrm{Sq}^{-1}, \mathrm{Sq}^0, \mathrm{Sq}^1, \ldots]$, so that $\mathcal{A}^{\mathrm{Big}}$ is the quotient of R by the ideal $I \subseteq R$ generated by the Adem relations.

For every pair of objects $M, N \in \mathcal{U}^{\mathrm{Big}}$, we let R act on $M \otimes N$ by the formula

$$\mathrm{Sq}^k(x \otimes y) = \sum_{k=k'+k''} \mathrm{Sq}^{k'}(x) \otimes \mathrm{Sq}^{k''}(y).$$

Observe that the sum appearing above is automatically finite, since $\mathrm{Sq}^{k'}(x) \otimes \mathrm{Sq}^{k''}(y)$ vanishes if $k' > \deg(x)$ or $k'' > \deg(y)$. The same argument shows that $M \otimes N$ is unstable, in the sense that $\mathrm{Sq}^k(x \otimes y) = 0$ for $k > \deg(x) + \deg(y)$.

We would like to prove the following:

Theorem 2. *For any pair of objects $M, N \in \mathcal{U}^{\mathrm{Big}}$, the tensor product $M \otimes N$ is again an unstable $\mathcal{A}^{\mathrm{Big}}$-module.*

In other words, we wish to show that the action of R on $M \otimes N$ factors through the quotient $R/I \simeq \mathcal{A}^{\mathrm{Big}}$. In other words, we wish to show that the submodule $I(M \otimes N) \subseteq M \otimes N$ vanishes. The submodule $I(M \otimes N)$ is generated by the submodules $I(x \otimes y) \subseteq M \otimes N$, where x and y are homogeneous elements of M and N. Let $m = \deg(x)$ and $n = \deg(y)$, so that x and y determine maps $\mathrm{F}^{\mathrm{Big}}(m) \to M$, $\mathrm{F}^{\mathrm{Big}}(n) \to N$. Here $\mathrm{F}^{\mathrm{Big}}(k)$ denotes the free unstable $\mathcal{A}^{\mathrm{Big}}$-module on a single generator $\overline{\nu}_k$ in degree k. The submodule $I(x \otimes y) \subseteq M \otimes N$ is a quotient of $I(\overline{\nu}_m \otimes \overline{\nu}_n) \subseteq \mathrm{F}^{\mathrm{Big}}(m) \otimes \mathrm{F}^{\mathrm{Big}}(n)$. It will therefore suffice to prove that this latter submodule vanishes.

For every integer k, let $\widetilde{\mathrm{F}^{\mathrm{Big}}}(k)$ denote the free R-module on a single generator $\widetilde{\nu}_k$, so that $\widetilde{\mathrm{F}^{\mathrm{Big}}}(k)$ has a basis consisting of expressions $\{\mathrm{Sq}^I\,\widetilde{\nu}_k\}$ where I ranges over *all* sequences of integers. We have canonical quotient maps

$$\widetilde{\mathrm{F}^{\mathrm{Big}}}(k) \to \mathrm{F}^{\mathrm{Big}}(k) \to F(k).$$

The construction of Definition 1 produces for us a map

$$\psi_{m,n} : \widetilde{\mathrm{F}^{\mathrm{Big}}}(m+n) \to \mathrm{F}^{\mathrm{Big}}(m) \otimes \mathrm{F}^{\mathrm{Big}}(n).$$

We wish to show that $\psi_{m,n}$ factors through $\mathrm{F}^{\mathrm{Big}}(m+n)$.

In a previous lecture, we defined a shift isomorphism

$$\widetilde{S} : \widetilde{\mathrm{F}^{\mathrm{Big}}}(k) \to \widetilde{\mathrm{F}^{\mathrm{Big}}}(k+1)$$

by the formula

$$\mathrm{Sq}^{i_k} \dots \mathrm{Sq}^{i_0}\,\widetilde{\nu}_k \mapsto \mathrm{Sq}^{i_k+2^k} \dots \mathrm{Sq}^{i_0+1}\,\widetilde{\nu}_{k+1}$$

and showed that $\widetilde{S}$ covers and isomorphism $S : \mathrm{F}^{\mathrm{Big}}(k) \to \mathrm{F}^{\mathrm{Big}}(k+1)$.

Suppose (for a contradiction) that there exists z in the kernel of the projection $\widetilde{\mathrm{F}^{\mathrm{Big}}}(m+n) \to \mathrm{F}^{\mathrm{Big}}(m+n)$ such that $\psi(z) \neq 0$. Then we can write $\psi(z)$ as a nontrivial linear combination $\sum \mathrm{Sq}^I\,\overline{\nu}_m \otimes \mathrm{Sq}^J\,\overline{\nu}_n$, where I and J range over (finitely many) admissible sequences of integers having excess $\leq m$ and $\leq n$, respectively. Consequently, for $p \gg 0$, we can write $(S \otimes S)^p(\psi z)$ as a nontrivial linear combination $\sum \mathrm{Sq}^{I'}\,\overline{\nu}_{m+p} \otimes \mathrm{Sq}^{J'}\,\overline{\nu}_{n+p}$, where the sequences I' and J' consist entirely of positive integers. It follows that the image of $\psi(z)$ under the composite map

$$\mathrm{F}^{\mathrm{Big}}(m) \otimes \mathrm{F}^{\mathrm{Big}}(n) \stackrel{S^p \otimes S^p}{\to} \mathrm{F}^{\mathrm{Big}}(m+p) \otimes \mathrm{F}^{\mathrm{Big}}(n+p) \to F(m+p) \otimes F(n+p)$$

is nonzero.

We now observe that the diagram

$$\begin{array}{ccc} \widetilde{\mathrm{F}^{\mathrm{Big}}}(m+n) & \xrightarrow{\psi_{m,n}} & \mathrm{F}^{\mathrm{Big}}(m) \otimes \mathrm{F}^{\mathrm{Big}}(n) \\ \downarrow{\scriptstyle \widetilde{S}^{2p}} & & \downarrow{\scriptstyle S^p \otimes S^p} \\ \widetilde{\mathrm{F}^{\mathrm{Big}}}(m+n+2p) & \xrightarrow{\psi_{m+p,n+p}} & \mathrm{F}^{\mathrm{Big}}(m+p) \otimes \mathrm{F}^{\mathrm{Big}}(m+p) \end{array}$$

commutes, where the horizontal arrows are defined as in Notation 1. Replacing z by $\widetilde{S}^{2p}(z)$ if necessary, we may assume that the composition

$$\widetilde{\mathrm{F}^{\mathrm{Big}}}(m+n) \stackrel{\psi_{m,n}}{\to} \mathrm{F}^{\mathrm{Big}}(m) \otimes \mathrm{F}^{\mathrm{Big}}(n) \to F(m) \otimes F(n)$$

does not vanish on z.

We have seen that there are injections $F(m) \hookrightarrow \mathrm{H}^*((\mathbf{R}P^\infty)^m)$ and $F(n) \hookrightarrow \mathrm{H}^*((\mathbf{R}P^\infty)^n)$. Amalgamating these, we obtain an injection $F(m) \otimes F(n) \hookrightarrow \mathrm{H}^*((\mathbf{R}P^\infty)^{m+n})$. Since the Cartan formula holds in $\mathrm{H}^*((\mathbf{R}P^\infty)^{m+n})$, the composite map

$$\phi : \widetilde{\mathrm{F}^{\mathrm{Big}}}(m+n) \stackrel{\psi_{m,n}}{\to} \mathrm{F}^{\mathrm{Big}}(m) \otimes \mathrm{F}^{\mathrm{Big}}(n) \to F(m) \otimes F(n) \hookrightarrow \mathrm{H}^*((\mathbf{R}P^\infty)^{m+n})$$

is simply the map of R-modules determined by the element $t_1 t_2 \dots t_{n+m} \in \mathrm{H}^{n+m}(\mathbf{R}P^\infty)^{m+n})$. Since $\mathrm{H}^*((\mathbf{R}P^\infty)^{m+n})$ satisfies the Adem relations, we have $\phi(z) = 0$, a contradiction. This completes the proof of Theorem 2.

It follows that the tensor product of Definition 1 determines a functor $\otimes : \mathcal{U}^{\mathrm{Big}} \times \mathcal{U}^{\mathrm{Big}} \to \mathcal{U}^{\mathrm{Big}}$. It is easy to see that this operation is commutative and associative, up to coherent isomorphism. In other words, it endows $\mathcal{U}^{\mathrm{Big}}$ with the structure of a symmetric monoidal category.

Corollary 3. *Let M and N be unstable modules over the Steenrod algebra $\mathcal{A}$. Then the tensor product $M \otimes N$ inherits the structure of an unstable module over the Steenrod algebra.*

Proof. We have seen that $M \otimes N$ has the structure of an unstable module over $\mathcal{A}^{\mathrm{Big}}$. To complete the proof, it will suffice to show that Sq^0 acts by the identity on $M \otimes N$. Unwinding the definition, we have

$$\mathrm{Sq}^0(x \otimes y) = \sum_k \mathrm{Sq}^k(x) \otimes \mathrm{Sq}^{-k}(y).$$

The right hand side vanishes if $k \neq 0$, and coincides with $x \otimes y$ when $k = 0$. □

The tensor product operation on the category of unstable Steenrod modules results from a comultiplicative structure which exists on the Steenrod algebra $\mathcal{A}$ itself:

Proposition 4. *There exists a ring homomorphism*

$$\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$

given by

$$\mathrm{Sq}^k \mapsto \sum_{k=k'+k''} \mathrm{Sq}^{k'} \otimes \mathrm{Sq}^{k''}.$$

Proof. The formula above evidently defines a ring homomorphism $\Delta : R \to \mathcal{A} \otimes \mathcal{A}$. Let K denote the kernel of the projection map $R \to \mathcal{A}$. It will suffice to show that $\Delta(K) = 0$. Suppose otherwise. Then there exists a nonzero element

$$T = \sum_\alpha \mathrm{Sq}^{I_\alpha} \otimes \mathrm{Sq}^{J_\alpha}$$

belonging to the image $\Delta(K)$, where (I_α, J_α) ranges over some finite set of admissible positive sequences. Choose a pair of positive integers (m, n) such that for some index α, m is at least as large as the excess of I_α and n is at least as large as the excess of J_α. Then we have $T(\nu_m \otimes \nu_n) \neq 0 \in F(m) \otimes F(n)$, which contradicts Corollary 3. □

The comultiplication $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ of Proposition 4 is in some respects simpler than the multiplication on $\mathcal{A}$: for example, it is commutative while the multiplication on $\mathcal{A}$ is not. We will return to this point in a future lecture.

We now introduce some terminology which we will need later.

Definition 5. An *unstable $\mathcal{A}^{Big}$-algebra* is an unstable $\mathcal{A}^{\mathrm{Big}}$-module M equipped with a commutative and associative multiplication $m : M \otimes M \to M$ satisfying the following conditions:

(1) The Cartan formula is satisfied:

$$\mathrm{Sq}^k(xy) = \sum_{k=k'+k''} \mathrm{Sq}^{k'}(x)\,\mathrm{Sq}^{k''}(y).$$

In other words, m is a map of $\mathcal{A}^{\mathrm{Big}}$-modules.

(2) For every homogeneous element $x \in M$, $\mathrm{Sq}^{\deg(x)}(x) = x^2$.

(3) M contains a unit element 1 satisfying

$$\mathrm{Sq}^i(1) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

An *unstable $\mathcal{A}$-algebra* is an unstable $\mathcal{A}^{\mathrm{Big}}$-algebra which is an $\mathcal{A}$-module: that is, an unstable $\mathcal{A}^{\mathrm{Big}}$-algebra M which satisfies $\mathrm{Sq}^0(x) = x$ for all $x \in M$.

Example 6. The cohomology $\mathrm{H}^*(X)$ of any space X has the structure of an unstable $\mathcal{A}$-algebra.

The cohomology $\mathrm{H}^*(A)$ of any E_∞-algebra over $\mathbf{F}_2$ has the structure of an unstable $\mathcal{A}^{\mathrm{Big}}$-algebra.

Our next goal is to understand the structure of free unstable algebras. For every integer n, we let $\mathrm{F}_{\mathrm{Alg}}(n)$ denote the free unstable $\mathcal{A}$-algebra generated by a single element μ_n of degree n, and $\mathrm{F}^{\mathrm{Big}}_{\mathrm{Alg}}(n)$ the free unstable $\mathcal{A}^{\mathrm{Big}}$-algebra generated by a single element $\overline{\mu}_n$ of degree n. We have an evident quotient map $\pi : \mathrm{F}^{\mathrm{Big}}_{\mathrm{Alg}}(n) \to \mathrm{F}_{\mathrm{Alg}}(n)$, uniquely determined by the requirement that $\pi(\overline{\mu}_n) = \mu_n$.

Let X denote the subspace of $\mathrm{F}^{\mathrm{Big}}_{\mathrm{Alg}}(n)$ spanned by the products

$$\{\mathrm{Sq}^{I_1}(\overline{\mu_n})\,\mathrm{Sq}^{I_2}(\overline{\mu_n})\ldots\mathrm{Sq}^{I_k}(\overline{\mu_n})\}.$$

Using relations (1) and (3), we deduce that X is a subalgebra of $\mathrm{F}^{\mathrm{Big}}_{\mathrm{Alg}}(n)$, so that $X = \mathrm{F}^{\mathrm{Big}}_{\mathrm{Alg}}(n)$. Moreover, relation (2) allows us to reduce any such monomial to a form where the sequences $I_1, \ldots, I_k$ are all distinct. Using the Adem relations and the instability condition, we can further reduce to considering such monomials where each sequence I_j is admissible and has excess $\leq n$. We have therefore proven half of the following result:

Theorem 7. (1) *The free unstable $\mathcal{A}^{Big}$-algebra $\mathrm{F}^{\mathrm{Big}}_{\mathrm{Alg}}(n)$ has a basis of monomials*

$$\{\mathrm{Sq}^{I_1}(\overline{\mu_n})\,\mathrm{Sq}^{I_2}(\overline{\mu_n})\ldots\mathrm{Sq}^{I_k}(\overline{\mu_n})\}$$

where $I_1 < \ldots < I_k$ (with respect to the lexicographical ordering, say) are admissible sequences of excess $\leq n$.

(2) *The free unstable $\mathcal{A}$-algebra $\mathrm{F}_{\mathrm{Alg}}(n)$ has a basis of monomials*

$$\{\mathrm{Sq}^{I_1}(\mu_n)\,\mathrm{Sq}^{I_2}(\mu_n)\ldots\mathrm{Sq}^{I_k}(\mu_n)\}$$

where $I_1 < \ldots < I_k$ are admissible positive sequences of excess $\leq n$.

The proof follows the same lines as our proof of the analogous fact for modules, and our construction of tensor products earlier in this lecture: we will reduce assertion (1) to assertion (2), using a shifting argument. Namely, there exists an isomorphism of algebras $\mathrm{F}^{\mathrm{Big}}_{\mathrm{Alg}}(n) \to \mathrm{F}^{\mathrm{Big}}_{\mathrm{Alg}}(n+1)$ given by the formula

$$(\mathrm{Sq}^{i^1_{j_1}}\ldots\mathrm{Sq}^{i^1_0}\,\overline{\mu}_n)\ldots(\mathrm{Sq}^{i^k_{j_k}}\ldots\mathrm{Sq}^{i^k_0}\,\overline{\mu}_n) \mapsto (\mathrm{Sq}^{i^1_{j_1}+2^{j_1}}\ldots\mathrm{Sq}^{i^1_0+1}\,\overline{\mu}_{n+1})\ldots(\mathrm{Sq}^{i^k_{j_k}+2^{j_k}}\ldots\mathrm{Sq}^{i^k_0+1}\,\overline{\mu}_{n+1}).$$

Consequently, any linear dependence among the expressions

$$M(I_1,\ldots,I_k) = \mathrm{Sq}^{I_1}(\overline{\mu_n})\,\mathrm{Sq}^{I_2}(\overline{\mu_n})\ldots\mathrm{Sq}^{I_k}(\overline{\mu_n}) \in \mathrm{F}^{\mathrm{Big}}_{\mathrm{Alg}}(n)$$

results in a linear dependence among analogous expressions $M(I'_1,\ldots,I'_k) \in \mathrm{F}^{\mathrm{Big}}_{\mathrm{Alg}}(n+p)$, for each $p \geq 0$. Choosing $p \gg 0$, we get a linear dependence involving monomials in which all of the sequences $(I'_1,\ldots,I'_k)$ are positive, which contradicts (2).

To prove (2), we need to produce some examples of unstable $\mathcal{A}$-algebras. We will return to this point in the next lecture.