18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

The Dual Steenrod Algebra (Lecture 13)

We have seen that the Steenrod algebra $\mathbf{A}$ admits a comultiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, described by the formula

$$\mathrm{Sq}^n \mapsto \sum_{n=n'+n''} \mathrm{Sq}^{n'} \otimes \mathrm{Sq}^{n''}.$$

This comultiplication map is obviously symmetric, and therefore endows the graded dual $\mathcal{A}^\vee = \oplus_n (\mathcal{A}^n)^\vee$ with the structure of a *commutative* ring. Our goal in this lecture is to understand the structure of $\mathcal{A}^\vee$.

For the remainder of this lecture, we will work in the category of (affine) schemes over the field $\mathbf{F}_2$. (In other words, we work in the opposite to the category of commutative $\mathbf{F}_2$-algebras.)

The noncommutative multiplication on $\mathcal{A}$ induces a *comultiplication* map $\mathcal{A}^\vee \to \mathcal{A}^\vee \otimes \mathcal{A}^\vee$, which in turn determines a map of $\mathbf{F}_2$-schemes

$$\operatorname{Spec} \mathcal{A}^\vee \times \operatorname{Spec} \mathcal{A}^\vee \to \operatorname{Spec} \mathcal{A}^\vee.$$

This map exhibits $\operatorname{Spec} \mathcal{A}^\vee$ as a *group scheme* over the field $\mathbf{F}_2$. Let us henceforth denote this group scheme by G.

For every topological space X, the Steenrod algebra acts on the cohomology ring $\mathrm{H}^*(X)$ via a map $\mathcal{A} \otimes \mathrm{H}^*(X) \to \mathrm{H}^*(X)$. If the cohomology ring $\mathrm{H}^*(X)$ is finite dimensional, then we can transpose this action to obtain a map

$$\mathrm{H}^*(X) \to \mathrm{H}^*(X) \otimes \mathcal{A}^\vee.$$

Rephrasing this in the language of algebraic geometry, we get a map

$$G \times \operatorname{Spec} \mathrm{H}^*(X) \to \operatorname{Spec} \mathrm{H}^*(X).$$

This map endows the scheme $\operatorname{Spec} \mathrm{H}^*(X)$ with an action of the group scheme G.

If $\mathrm{H}^*(X)$ is not finite-dimensional, then we need to be a bit more careful. Suppose instead that $\mathrm{H}^*(X)$ is finite dimensional in each degree. For each $n \geq 0$, the direct sum $R_n = \oplus_{0 \leq k \leq n} \mathrm{H}^k(X)$ can be viewed as a quotient of the cohomology ring $\mathrm{H}^*(X)$, and inherits the structure of an unstable $\mathcal{A}$-algebra. Using the above argument, we obtain an action

$$G \times \operatorname{Spec} R_n \to \operatorname{Spec} R_n.$$

Moreover, if $n = 1$, then this action is trivial.

Let us now specialize to the case where X is the space $\mathbf{R}P^\infty$. In this case, the cohomology ring $\mathrm{H}^*(X)$ is isomorphic to $\mathbf{F}_2[t]$. We therefore have isomorphisms $R_n \simeq \mathbf{F}_2[t]/(t^{n+1})$ for $n \geq 0$. For each $n \geq 0$, there exists a group scheme parametrizing automorphisms of $\operatorname{Spec} R_n$ which induce the identity on $\operatorname{Spec} R_1$. We will denote this group scheme by H_n. By definition, H_n has the following universal property:

$$\operatorname{Hom}(\operatorname{Spec} B, H_n) \simeq \operatorname{Hom}^0(\operatorname{Spec} B \times \operatorname{Spec} R_n, \operatorname{Spec} R_n) \simeq \operatorname{Hom}^0(\mathbf{F}_2[t]/(t^{n+1}, B[t]/(t^{n+1}) \simeq t + t^2 B/(t^{n+1}B),$$

(here the superscripts indicate the requirement that the morphism reduce to the identity on R_1) so H_n is just isomorphic to an $(n-1)$-dimensional affine space $\mathbf{A}^n$. Let H_∞ denote the inverse limit of the tower

$$\ldots \to H_2 \to H_1 \to H_0,$$

so that H_∞ is the infinite dimensional affine space which is the automorphism group of the formal scheme $\operatorname{Spf} \mathbf{F}_2[[t]]$. More concretely, we are just saying that every automorphism of the power series ring $B[[t]]$ which reduces to the identity modulo t^2 is given by a transformation

$$t \mapsto t + b_1 t^2 + b_2 t^3 + \ldots,$$

so we get an identification $H_\infty \simeq \operatorname{Spec} \mathbf{F}_2[b_1, b_2, \ldots]$

The above analysis gives us a map of group schemes $\phi : G \to H_\infty$. Our first result is:

Proposition 1. *The map $\phi : G \to H_\infty$ is a monomorphism.*

To prove this, let $G_0 \subseteq G$ be the kernel of the homomorphism ϕ. Then G_0 acts trivially on the formal spectrum $\operatorname{Spf} \mathrm{H}^*(\mathbf{R}P^\infty)$. It follows that the diagonal action of G_0 on

$$\operatorname{Spf} \mathrm{H}^*(\mathbf{R}P^\infty) \times \ldots \times \operatorname{Spf} \mathrm{H}^*(\mathbf{R}P^\infty) \simeq \operatorname{Spf} \mathrm{H}^*((\mathbf{R}P^\infty)^k)$$

is trivial for all k.

We observe that $G_0 = \operatorname{Spec} C$, where C is some Hopf algebra quotient of the dual Steenrod algebra $\mathcal{A}^\vee$. It is not difficult to see that C inherits a grading from $\mathcal{A}^\vee$, so that the graded dual $C^\vee$ can be identified with a subalgebra of the Steenrod algebra $\mathcal{A}$. The above analysis shows that $C^\vee$ acts trivially on the cohomology $\mathrm{H}^*((\mathbf{R}P^\infty)^k)$ for all $k \geq 0$. We claim that $C^\vee \simeq \mathbf{F}_2$. If not, then we can find some nonconstant element of $C^\vee$ of the form $\sum_\alpha \mathrm{Sq}^{I_\alpha}$, where I_α ranges over some collection of admissible positive sequences. Choosing k larger than the excess of each I_α, we see that $C^\vee$ acts nontrivially on $t_1 \ldots t_k \in \mathrm{H}^k((\mathbf{R}P^\infty)^k)$, a contradiction. Thus $C^\vee \simeq \mathbf{F}_2$, so $G_0 \simeq \operatorname{Spec} \mathbf{F}_2$ and the map ϕ is a monomorphism as desired.

We now wish to describe the image of the map ϕ. For this, we observe that the formal affine line $\hat{\mathbf{A}}^1 \simeq \operatorname{Spf} \mathbf{F}_2[[t]]$ is isomorphic to the *formal additive group* over the field $\mathbf{F}_2$. In other words, we have an addition map

$$\hat{\mathbf{A}}^1 \times \hat{\mathbf{A}}^1 \to \hat{\mathbf{A}}^1,$$

which is described in coordinates by the map of power series rings

$$\mathbf{F}_2[[t]] \to \mathbf{F}_2[[t_1, t_2]]$$

$$t- > t_1 + t_2.$$

In fact, this map comes from topology. The group Σ_2 is abelian, so the multiplication map

$$\Sigma_2 \times \Sigma_2 \to \Sigma_2$$

is a group homomorphism. It follows that we obtain a map of classifying spaces

$$B\Sigma_2 \times B\Sigma_2 \simeq B(\Sigma_2 \times \Sigma_2) \to B\Sigma_2.$$

The induced map on cohomology

$$\mathrm{H}^*(\mathbf{R}P^\infty) \to \mathrm{H}^*(\mathbf{R}P^\infty \times \mathbf{R}P^\infty)$$

is also described by the formula

$$t \mapsto t_1 + t_2.$$

It follows that the action of the Steenrod algebra $\mathcal{A}$ is compatible with the comultiplication on $\mathrm{H}^*(\mathbf{R}P^\infty)$. In other words, the action of the group scheme $G = \operatorname{Spec} \mathcal{A}^\vee$ on the formal affine line $\hat{\mathbf{A}}^1$ preserves the group structure on $\hat{\mathbf{A}}^1$.

Let $\operatorname{End}(\mathbf{A}^1)$ denote the subgroup scheme of H_∞ which preserves the group structure on $\mathbf{A}^1$. We note that a B-valued point of H_∞ is an automorphism of $B[[t]]$ of the form

$$t \mapsto t + b_1 t^2 + b_2 t^3 + \ldots.$$

This B-valued point belong to $\mathrm{End}(\mathbf{A}^1)$ if and only if the power series $f(t) = t + b_1 t^2 + b_2 t^3 + \ldots$ is additive, in the sense that $f(t_1 + t_2) = f(t_1) + f(t_2) \in B[[t_1, t_2]]$. Since we are working in characteristic 2, additivity is equivalent to the requirement that the terms $b_{i-1} t^i$ vanish unless i is a power of 2. In other words, we can identify $\mathrm{End}(\mathbf{A}^1)$ with the infinite dimensional affine space parametrizing power series of the form

$$t + b_1 t^2 + b_3 t^4 + b_7 t^8 + \ldots.$$

Theorem 2. *The map ϕ induces an isomorphism $G \to \mathrm{End}(\mathbf{A}^1)$.*

In other words, we claim that the corresponding map of commutative rings

$$\psi : \mathbf{F}_2[b_1, b_3, b_7, \ldots] \to \mathcal{A}^\vee$$

is an isomorphism. Proposition 1 implies that ψ is surjective. Moreover, ψ is a map of graded rings, where each b_i is regarded as having degree i. It will therefore suffice to show that the algebras $\mathbf{F}_2[b_1, b_3, b_7, \ldots]$ and $\mathcal{A}^\vee$ have the same dimensions in each degree.

Fix an integer $n \geq 0$. The nth graded piece of $\mathbf{F}_2[b_1, b_3, b_7, \ldots]$ is spanned by monomials

$$b_1^{\epsilon_1} b_3^{\epsilon_2} b_7^{\epsilon_3} \ldots,$$

which are indexed by sequences of nonnegative integers $(\epsilon_1, \epsilon_2, \ldots)$ satisfying $\sum_k (2^k - 1)\epsilon_k = n$.

We have also seen that the the Steenrod algebra $\mathcal{A}$ has a basis consisting of expressions $\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \ldots \mathrm{Sq}^{i_m}$, where the quantities

$$\delta_k = \begin{cases} i_k - 2i_{k+1} & \text{if } k < m \\ i_m & \text{if } k = m \\ 0 & \text{if } k > m \end{cases}$$

are required to be nonnegative. Moreover, we have

$$i_k = \delta_k + 2\delta_{k+1} + 4\delta_{k+2} + \ldots$$

so that the total degree of Sq^I is

$$\sum_{k>0} i_k = \sum_{k>0, m\geq 0} \delta_{k+m} 2^m = \sum_{k'>0} \delta_{k'} (2^{k'} - 1).$$

We therefore obtain a bijection from a basis of $\mathbf{F}_2[b_1, b_3, \ldots]^n$ to a basis of $\mathcal{A}^n$, given by the correspondence

$$(\epsilon_1, \epsilon_2, \ldots) \leftrightarrow (\delta_1, \delta_2, \delta_3, \ldots).$$

Remark 3. In fact, more is true: the bijection described above is actually upper-triangular with respect to duality between $\mathcal{A}$ and $\mathbf{F}_2[b_1, b_3, \ldots]$ determined by the ring homomorphism ψ. This is implicit in our proof that the admissible monomials are linearly independent in $\mathcal{A}$.

Corollary 4. *The dual Steenrod algebra $\mathcal{A}^\vee$ is isomorphic to a polynomial ring $\mathbf{F}_2[b_1, b_3, b_7, \ldots]$.*

We can describe the comultiplication on $\mathcal{A}^\vee$ (and therefore the multiplication on $\mathcal{A}$) very concretely in terms of the isomorphism of Corollary 4. This comultiplication correpsonds to the group structure on $\mathrm{End}(\mathbf{A}^1)$: in other words, it corresponds to composition of transformations having the form $t \mapsto t + b_1 t^2 + b_3 t^4 + \ldots$. Let $f(t) = \sum_{i \geq 0} b_{2^i - 1} t^{2^i}$ and $g(t) = \sum_{j \geq 0} b'_{2^j - 1} t^{2^j}$. Then

$$(f \circ g)(t) = \sum_{i,j \geq 0} b_{2^i - 1} (b'_{2^j - 1})^{2^i} t^{2^{i+j}}.$$

Consequently, the comultiplication on the ring $\mathbf{F}_2[b_1, b_3, \ldots]$ can be described by the formula

$$b_{2^k-1} \mapsto \sum_{k=i+j} b_{2^i-1} \otimes b_{2^j-1}^{2^i}.$$

Here we include the extreme possibilities $i = 0$ and $j = 0$, in which case we agree to the convention that $b_0 = 1 \in \mathbf{F}_2[b_1, b_3, \ldots]$.

Remark 5. The results above describe the dual Steenrod algebra $\mathcal{A}^\vee$ as the algebra of functions on the algebraic group $G \simeq \mathrm{End}(\mathbf{A}^1)$. We get a dual description of the Steenrod algebra $\mathcal{A}$ itself as an algebra of *distributions* on the group G: namely, $\mathcal{A}$ is isomorphic to the space of distributions on G which are set-theoretically supported at the identity. In this language, the (noncommutative) multiplication on $\mathcal{A}$ is induced by convolution.