18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

The Frobenius (Lecture 14)

Our goal in this lecture is to study some of the basic features of the category $\mathcal{U}$ of unstable modules over the Steenrod algebra $\mathcal{A}$. We begin with a few general remarks.

For every commutative algebra R over the field $\mathbf{F}_2$, there is a canonical ring homomorphism $F : R \to R$, called the *Frobenius morphism*, given by $F(x) = x^2$. The Frobenius map is functorial with respect to all homomorphisms between commutative $\mathbf{F}_2$-algebras: in other words, every map $f : R \to R'$ fits into a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ R & \xrightarrow{f} & R'. \end{array}$$

In particular, if R is a commutative Hopf algebra over $\mathbf{F}_2$ with comultiplication $\Delta : R \to R \otimes R$, then we have a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\Delta} & R \otimes R \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F\otimes F} \\ R & \xrightarrow{\Delta} & R \otimes R. \end{array}$$

In other words, the Frobenius map F is a homomorphism of Hopf algebras.

We apply this remark in the case where R is the dual Steenrod algebra $\mathcal{A}^\vee \simeq \mathbf{F}_2[b_1, b_3, b_7, \ldots]$. We have a map of Hopf algebras

$$F : \mathcal{A}^\vee \to \mathcal{A}^\vee.$$

Passing to the graded dual, we obtain another map of Hopf algebras $V : \mathcal{A} \to \mathcal{A}$, called the *Verschiebung*.

Let us compute the map V. Since V is a map of algebras, it will suffice to compute $V(\mathrm{Sq}^n)$ for each $n \geq 0$. Let $\langle , \rangle : \mathcal{A} \otimes \mathcal{A}^\vee \to \mathbf{F}_2$ denote the pairing between the Steenrod algebra and its dual. By definition, we have

$$\langle V(\mathrm{Sq}^n), x\rangle = \langle \mathrm{Sq}^n, x^2\rangle.$$

Since the algebra structure on $\mathcal{A}^\vee$ is dual to the comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, we get

$$\langle \mathrm{Sq}^n, x^2\rangle = \langle \Delta \mathrm{Sq}^n, x \otimes x\rangle = \sum_{n=i+j} \langle \mathrm{Sq}^i, x\rangle\langle \mathrm{Sq}^j, x\rangle.$$

We note that the terms in this sum for which $i \neq j$ cancel in pairs. Moreover, if n is even, the term with $i = j = \frac{n}{2}$ coincides with

$$\langle \mathrm{Sq}^i, x\rangle\langle \mathrm{Sq}^j, x\rangle = \langle \mathrm{Sq}^{\frac{n}{2}}, x\rangle.$$

We can summarize this calculation as follows:

Proposition 1. *The Vershiebung map $V : \mathcal{A} \to \mathcal{A}$ is given by the formula*

$$V(\mathrm{Sq}^n) = \begin{cases} \mathrm{Sq}^{\frac{n}{2}} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Remark 2. We could instead regard V as being *defined* by the formula of Proposition 1. Then we would need to check that V is well-defined, which is an exercise in manipulating the Adem relations.

Let M be a module over the Steenrod algebra $\mathcal{A}$, so that we have a ring homomorphism $\mathcal{A} \to \mathrm{End}(M)$. Composing with the Vershiebung map $V : \mathcal{A} \to \mathcal{A}$, we get a new homomorphism $\mathcal{A} \to \mathrm{End}(M)$, which gives a new $\mathcal{A}$-module structure on M. We will denote this new $\mathcal{A}$-module by ΦM. More concretely:

(1) The elements of ΦM can be identified with the elements of M. When it is important to distinguish between the M and ΦM, we let $\Phi(x)$ denote the element of ΦM corresponding to $x \in M$.

(2) The Steenrod algebra acts on ΦM by the formula

$$\mathrm{Sq}^n \Phi(x) = \begin{cases} \Phi(\mathrm{Sq}^{\frac{n}{2}} x) & \text{n even} \\ 0 & \text{n odd.} \end{cases}$$

The map V does not preserve the grading on the Steenrod algebra $\mathcal{A}$: we have instead $\deg V(a) = \frac{\deg(a)}{2}$ if a is homogeneous of even degree (and $V(a)$ vanishes if a has odd degree). If M is a graded $\mathcal{A}$-module, then ΦM again has the structure of a graded $\mathcal{A}$-module via the following convention:

(3) For each $n \geq 0$, we let

$$(\Phi M)^n = \begin{cases} M^{\frac{n}{2}} & \text{n even} \\ 0 & \text{n odd.} \end{cases}$$

Note that if M is an unstable $\mathcal{A}$-module, then ΦM is again unstable: if x is a nonzero element of $(\Phi M)^n$, then $n = 2k$ is even and $x = \Phi(x_0)$ for some $x_0 \in M^k$. If $m > n$, then $\mathrm{Sq}^m(x)$ vanishes by definition if m is odd, and is equal to $\Phi(\mathrm{Sq}^{\frac{m}{2}}(x_0))$ if m is even; this will also vanish since $\frac{m}{2} > k$ and M is assumed to be unstable.

Proposition 3. *Let M be an unstable module over the Steenrod algebra $\mathcal{A}$. Then there is a canonical homomorphism of $\mathcal{A}$-modules*

$$f : \Phi M \to M$$

defined by the formula $f(\Phi(x)) = \Phi(\mathrm{Sq}^{\deg x}(x))$ when x is homogeneous.

Remark 4. If M is an unstable algebra over $\mathcal{A}$, we can rewrite the definition of f as $f(\Phi(x)) = \Phi(x^2)$. In other words, we can think of f as a kind of Frobenius map.

Proof. We must show that f is compatible with the action of the Steenrod algebra: in other words, we must show that for every homogeneous element x, we have

$$f(\mathrm{Sq}^n \Phi(x)) = \mathrm{Sq}^n \mathrm{Sq}^{\deg x}(x).$$

There are three cases to consider.

If $n > 2\deg(x)$, then both sides vanish in view our our assumption that M is unstable. If $n = 2\deg(x)$, then we have

$$f(\mathrm{Sq}^n \Phi(x)) = f(\Phi(\mathrm{Sq}^{\deg x}(x)) = \mathrm{Sq}^{2\deg x} \mathrm{Sq}^{\deg x} x = \mathrm{Sq}^n \mathrm{Sq}^{\deg x} x.$$

If $n < 2\deg(x)$, then we can rewrite $\mathrm{Sq}^n \mathrm{Sq}^{\deg x}(x)$ using the Adem relations. We get

$$\mathrm{Sq}^n \mathrm{Sq}^{\deg x}(x) = \sum_k (2k - n, \deg(x) - k - 1) \mathrm{Sq}^{\deg(x)+k} \mathrm{Sq}^{n-k}(x).$$

Terms with $2k > n$ vanish since $\deg(x) + k > \deg(\mathrm{Sq}^{n-k}(x)) = n - k + \deg(x)$. Terms with $2k < n$ vanish since $2k - n < 0$. We therefore have

$$\mathrm{Sq}^n \mathrm{Sq}^{\deg x}(x) = \begin{cases} 0 & \text{n odd} \\ \mathrm{Sq}^{\deg x + \frac{n}{2}} \mathrm{Sq}^{\frac{n}{2}} x & \text{n even.} \end{cases}$$

On the other hand, we have

$$\mathrm{Sq}^n(\Phi(x)) = \begin{cases} 0 & \text{n odd} \\ \Phi(\mathrm{Sq}^{\frac{n}{2}}) & \text{n even.} \end{cases}$$

and in the latter case $\deg(\mathrm{Sq}^{\frac{n}{2}}(x)) = \deg(x) + \frac{n}{2}$, so the desired equality holds. □

Let us study the behavior of the homomorphism f in the case where $M = F(n)$ is the free unstable $\mathcal{A}$-module on one generator ν_n. In this case, M has a basis $\{\mathrm{Sq}^I \nu_n\}$, where I ranges over admissible positive sequences $(i_1, \ldots, i_k)$ of excess $\leq n$. We observe that $f(\Phi(\mathrm{Sq}^I \nu_n)) = \mathrm{Sq}^{I'} \nu_n$, where I' is the sequence $(i_0, i_1, i_2, \ldots, i_k)$ with

$$i_0 = \deg \mathrm{Sq}^I \nu_n = i_1 + i_2 + \ldots + i_k + n.$$

In particular, the excess $i_0 - i_1 - \ldots - i_k$ of I' is precisely n. Conversely, if I' is an admissible sequence of excess n, then $I' = (\deg \mathrm{Sq}^I \nu_n, i_1, i_2, \ldots, i_k)$, where $I = (i_1, i_2, \ldots, i_k)$. In other words:

Proposition 5. *The map $f : \Phi F(n) \to F(n)$ is injective, and its image is spanned by expressions $\{\mathrm{Sq}^I \nu_n\}$ where I is positive, admissible, and has excess exactly n.*

The cokernel of the map $f : \Phi F(n) \to F(n)$ has a basis given by the images of the expressions $\{\mathrm{Sq}^I \nu_n\}$, where I ranges over admissible positive sequences of excess $< n$. Up to a change of grading, this is identical to the structure of the free unstable module $F(n-1)$. In order to describe the situation more systematically, we introduce the following definition:

Definition 6. Let M be an unstable module over the Steenrod algebra $\mathcal{A}$. We define a new unstable $\mathcal{A}$-module ΣM as follows:

(1) As a vector space, $\Sigma M \simeq M$, and this isomorphism is compatible with the action of the Steenrod algebra.

(2) The grading on ΣM is defined by the formula $(\Omega M)^n \simeq M^{n-1}$.

In other words, ΣM is the module $M \otimes \mathbf{F}_2[-1]$, where $\mathbf{F}_2[-1]$ denotes a single copy of $\mathbf{F}_2$ in degree 1 (with its unique $\mathcal{A}$-module structure).

Warning 7. The notation introduced in Definition 6 is incompatible with our notation for suspensions of complexes used in previous lectures: if V is a complex with a good symmetric multiplication, we have an isomorphism of $\mathcal{A}$-modules $H^*(\Omega V) = \Sigma H^*(V)$.

If M is an unstable $\mathcal{A}$-module, then ΣM is again unstable. However, Σ does not define an equivalence from the category $\mathfrak{U}$ to itself, because the obvious "inverse" construction does not preserve instability. For each unstable $\mathcal{A}$-module M, let $\overline{\Omega} M$ denote the $\mathcal{A}$-module M, with the grading $(\overline{\Omega} M)^n \simeq M^{n+1}$. Then $\overline{\Omega} M$ is not necessarily unstable: an element $x \in (\overline{\Omega} M)^n$ can be identified with an element $x \in M^{n+1}$, so that x need not be annihilated by Sq^{n+1}. However, we can correct this deficiency by passing to a quotient: let ΩM denote the quotient of $\overline{\Sigma M}$ by the submodule generated by $\mathrm{Sq}^k x$ for $k > n$, $x \in (\overline{\Sigma} M)^n$. (In fact, it suffices to take $k = n + 1$ here). Then the construction $M \mapsto \Omega M$ defines a functor from the category of unstable $\mathcal{A}$-modules to itself, and this construction is left adjoint to the functor Σ.

We observe that, for every unstable $\mathcal{A}$-module M, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(\Omega F(n), M) \simeq \mathrm{Hom}_{\mathcal{A}}(F(n), \Sigma M) \simeq (\Sigma M)^n \simeq M^{n-1}.$$

Conequently, we can identify $\Omega F(n)$ with $F(n-1)$ as an unstable $\mathcal{A}$-module. The adjoint of this identification is a map $F(n) \to \Sigma F(n-1)$. We can restate Proposition 5 as follows:

Proposition 8. *For each $n > 0$, we have a short exact sequence*

$$0 \to \Phi F(n) \xrightarrow{f} F(n) \xrightarrow{u} \Sigma\Omega F(n) \to 0$$

where u is the unit map for the adjunction between Ω and Σ and f is the map of Proposition 3.

Proposition 8 admits a generalization where we replace $F(n)$ by an arbitrary unstable $\mathcal{A}$-module M. We observe that the functors $M \mapsto \Phi M$ and $M \to \Sigma M$ are obviously exact. However, the functor $M \mapsto \Omega M$ is only right exact. We can therefore define left-derived functors $L^i\Omega M$ to be the homologies of the complex

$$\ldots \to \Omega P_2 \to \Omega P_1 \to \Omega P_0 \to 0,$$

where $\ldots \to P_1 \to P_0 \to M$ is a resolution of M by free unstable $\mathcal{A}$-modules. A standard argument in homological algebra shows that this definition is independent of the choice of resolution, up to canonical isomorphism.

Theorem 9. *For every unstable $\mathcal{A}$-module M, there is a canonical exact sequence*

$$0 \to \Sigma L^1\Omega M \to \Phi M \xrightarrow{f} M \xrightarrow{u} \Sigma\Omega M \to 0$$

where u is the unit map for the adjunction between Σ and Ω, and f is the map described in Proposition 3. Moreover, the derived functors $L^i\Omega$ vanish for $i > 1$.

Proof. Choose a free resolution $P_\bullet$ of M. Using Proposition 8, we get a short exact sequence of complexes

$$0 \to \Phi P_\bullet \to P_\bullet \to \Sigma\Omega P_\bullet \to 0.$$

The desired result now follows from the associated long exact sequence, since the complexes $\Phi P_\bullet$ and $P_\bullet$ are exact in degrees > 0. □