18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Finiteness Conditions (Lecture 15)

Our goal in this lecture is to prove that the category $\mathcal{U}$ of unstable $\mathcal{A}$-modules is locally Noetherian. We begin with by recalling a few definitions.

Definition 1. An object X of a Grothendieck abelian category $\mathcal{C}$ is *Noetherian* if every ascending chain of subobjects

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$$

eventually stabilizes.

We will say that a Grothendieck abelian category $\mathcal{C}$ is *locally Noetherian* if every object $X \in \mathcal{C}$ is the direct limit of its Noetherian subobjects. direct limit

Remark 2. Suppose given an exact sequence

$$0 \to X' \to X \to X'' \to 0$$

in a Grothendieck abelian category $\mathcal{C}$. Then X is Noetherian if and only if X' and X'' are Noetherian. The "only if" direction is clear: any infinite ascending sequence of subobjects of X' or X'' gives rise to an infinite ascending sequence of subobjects of X. For the converse, we observe that an infinite ascending sequence of objects

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots \subseteq X$$

gives rise to a collection of long exact sequences

$$0 \to X_i \cap X' \to X_i \to (\operatorname{Im} X_i \to X'') \to 0.$$

If X' and X'' are Noetherian, then the subobjects $X_i \cap X'$ and $\operatorname{Im} X_i \to X''$ are independent of i for $i \gg 0$, so that X_i is also independent of i for $i \gg 0$.

In particular, the collection of Noetherian objects of $\mathcal{C}$ is closed under finite direct sums.

Example 3. Let R be a (left) Noetherian ring. Then the category $\mathcal{C}$ of (left) R-modules is locally Noetherian. An object $X \in \mathcal{C}$ is Noetherian if and only if it is finitely generated as an R-module.

The Steenrod algebra $\mathcal{A}$ itself is *not* left Noetherian. For example, the left ideal of $\mathcal{A}$ generated by $\{\mathrm{Sq}^i\}_{i>0}$ is not finitely generated. Nevertheless, we have the following analogue of Example 3:

Theorem 4. (1) *The category $\mathcal{U}$ of unstable $\mathcal{A}$-algebras is locally Noetherian.*

(2) *An object $M \in \mathcal{U}$ is Noetherian if and only if it is finitely generated as a $\mathcal{A}$-module.*

The implication (2) $\Rightarrow$ (1) is clear, since every object in $\mathcal{U}$ is the direct limit of its finitely generated subobjects. The "only if" direction follows formally from the following observation:

Lemma 5. *An object $M \in \mathcal{U}$ is Noetherian if and only if every submodule $M' \subseteq M$ is finitely generated.*

Proof. If $M' \subseteq M$ is not finitely generated, then we can find an infinite ascending sequence of submodules

$$\mathcal{A}\, x_1 \subset \mathcal{A}\, x_1 + \mathcal{A}\, x_2 \subset \ldots \subseteq M'$$

by choosing each x_i to be an element of M' which does not belong to the submodule generated by $\{x_j\}_{j<i}$.

Conversely, if M is not Noetherian, we can find an infinite ascending sequence of submodules

$$M_0 \subset M_1 \subset M_2 \subset \ldots$$

Let $M' = \bigcup M_i \subseteq M$. Then M' cannot be finitely generated: if it were, then it would be generated by elements belonging to M_n for $n \gg 0$, so that $M_{n+1} \subseteq M_n$, contrary to our assumption. □

We wish to prove that *every* finitely generated unstable $\mathcal{A}$-module M is Noetherian. In this case, we can write M as a quotient of a finite sum $\oplus_i F(n_i)$. Remark 2 implies that the collection of Noetherian objects of $\mathcal{U}$ is stable under finite direct sums and quotients. In view of Lemma 5, it will suffice to prove the following:

Theorem 6. *Let $F(n)$ denote the free unstable $\mathcal{A}$-module on a single generator ν_n in degree n. Then every submodule $M \subseteq F(n)$ is finitely generated.*

We will prove Theorem 6 using induction on n. The case $n = 0$ is obvious. To handle the general case, we will need the following:

Lemma 7. *Let M be an unstable $\mathcal{A}$-module. If ΩM is finitely generated and M^0 is finitely generated, then M is finitely generated.*

Proof. If ΩM is finitely generated, then $\Sigma\Omega M$ is finitely generated. In the last lecture, we saw that there is an exact sequence

$$\Phi M \to M \to \Sigma\Omega M \to 0.$$

Choose a finite set of (homogeneous) generators $\{\overline{x}_i\}$ for $\Sigma\Omega M$, and lift them to (homogeneous) elements $\{x_i \in M\}$. Let N be the submodule of M generated by M^0 and $\{x_i\}$. We claim that $N = M$. We will prove by induction that $N^n = M^n$ for all integers n. If $n = 0$ there is nothing to prove. If n is odd, then the exact sequence above gives $M^n \simeq (\Sigma\Omega M)^n$, and the result is obvious. If $n = 2k > 0$ is even, then our exact sequence can be rewritten

$$M^k \stackrel{\mathrm{Sq}^k}{\to} M^{2k} \to (\Sigma\Omega M)^{2k} \to 0.$$

It is clear that M^{2k} is generated by N^{2k} together with the image of Sq^k. The inductive hypothesis guarantees that $\mathrm{Sq}^k M^k = \mathrm{Sq}^k N^k \subseteq N^{2k}$, so that $M^{2k} = N^{2k}$ as desired. □

We are now ready to proceed with the proof of Theorem 6.

We define an ascending chain of submodules

$$M = M_0 \subseteq M_1 \subseteq \ldots \subseteq F(n)$$

as follows: let M_n be defined so that $\Phi^n M_n$ is the inverse image of $M = M_0$ under the iterated Frobenius map

$$\Phi^n F(n) \to \Phi^{n-1} F(n) \to \ldots \to F(n).$$

We have for each $m \geq 0$ an exact sequence

$$\Phi M_{m+1} \to M_m \to M'_m \to 0,$$

where M'_m denotes the image of M_m in $\Sigma\Omega F(n) \simeq \Sigma F(n-1)$. The inductive hypothesis implies that every ascending sequence of submodules of $F(n-1)$ stabilizes, so that $M'_m = M'_{m+1}$ for $m \geq m_0$.

We claim also that $M_m = M_{m+1}$ for $m \geq m_0$. To prove this, we show by induction on k that the sequence

$$M^k_{m_0} \subseteq M^k_{m_0+1} \subseteq M^k_{m_0+2} \subseteq \ldots$$

is constant. If $k=0$ there is nothing to prove. For $k>0$, we have exact sequences

$$M_{m+1}^{\frac{k}{2}} \xrightarrow{\mathrm{Sq}^{\frac{k}{2}}} M_m^k \to {M'}_m^k \to 0$$

(here the left term vanishes by convention if k is odd). The desired result follows from the inductive hypothesis (since $\frac{k}{2} < k$).

We now prove that each M_m is finitely generated, using descending induction on m. We observe that $\Sigma\Omega M_{m_0} \simeq M'_{m_0}$ is a submodule of $\Sigma F(n-1)$, and therefore finitely generated by our inductive hypothesis. Therefore M_{m_0} is finitely generated by Lemma 7.

To handle the general case, we use the exact sequence

$$\Phi M_{m+1} \to M_m \to M'_m \to 0.$$

The inductive hypothesis guarantees that M_{m+1} is finitely generated. Let $\{x_i\}$ be a finite set of generators for M_{m+1}. Then $\{\Phi(x_i)\}$ is a finite set of generators for ΦM_{m+1}. Let $\{y_i\}$ denote the images of these generators in M_m. Since M'_m is a submodule of $\Sigma F(n-1)$, we deduce that M'_m is generated by a finite set of elements $\{\overline{z}_j\}$. Choose elements $\{z_j\}$ in M_m which lift these elements. It is now clear that M_m is generated by the finite set $\{y_i\} \cup \{z_j\}$. This completes the proof of Theorem 6.

Our next goal in this lecture is to prove the following result:

Proposition 8. *The collection of finitely generated unstable $\mathcal{A}$-modules is closed under the formation of tensor products.*

In other words, we wish to show that if M and N are finitely generated, then $M \otimes N$ is finitely generated. We can write M as a quotient some finite sum $\oplus_i F(m_i)$, so that $M \otimes N$ is a quotient of some finite sum $\oplus_i (F(m_i) \otimes N)$. It will therefore suffice to show that each $F(m_i) \times N$ is finitely generated. Applying the same argument to N, we are reduced to proving the following special case of Proposition 8:

Proposition 9. *For every pair of nonnegative integers $m, n \geq 0$, the tensor product $F(m) \otimes F(n)$ is finitely generated.*

To prove Proposition 9, we first recall the structure of the free unstable $\mathcal{A}$-module $F(n)$. Let X denote a product of n copies of $\mathbf{R}P^\infty$, so that $\mathrm{H}^*(X) \simeq \mathbf{F}_2[t_1, t_2, \ldots, t_n]$. Then we can identify $F(n)$ with the $\mathcal{A}$-submodule of $\mathrm{H}^*(X)$ generated by the element $t_1 \ldots t_n \in \mathrm{H}^n(X)$. Moreover, we have an explicit description of this submodule: it consists of those polynomials $f(t_1, \ldots, t_n)$ which are symmetric and whose exponents involve only powers of 2. In particular, $F(1)$ can be identified with the $\mathcal{A}$-module of $\mathbf{F}_2[t]$ spanned by $\{t, t^2, t^4, \ldots\}$. We can therefore identify $F(n)$ with the submodule of $F(1)^{\otimes n}$ spanned by the symmetric polynomials: in other words, we have an isomorphism

$$F(n) \simeq (F(1)^{\otimes n})^{\Sigma_n} \subseteq F(1)^{\otimes n}.$$

Let us turn to the proof of Proposition 9. We have an inclusion

$$F(m) \otimes F(n) \subseteq (F(1)^{\otimes m}) \otimes (F(1)^{\otimes n}) \simeq F(1)^{\otimes m+n}.$$

Since the collection of finitely generated unstable $\mathcal{A}$-modules is closed under the formation of subobjects, it will suffice to prove the following:

Proposition 10. *For each $n \geq 0$, the $\mathcal{A}$-module $F(1)^{\otimes n}$ is finitely generated.*

The proof goes by induction on n, the case $n=0$ being obvious. To handle the general case, we use Lemma 7: it will suffice to show that $\Sigma\Omega F(1)^{\otimes n}$ is finitely generated. We observe that $F(1)^{\otimes n}$ can be identified with the submodule of $\mathbf{F}_2[t_1, \ldots, t_n]$ spanned by monomials of the form $t_1^{2^{b_1}} \ldots t_n^{2^{b_n}}$. We have an exact sequence

$$\Phi F(1)^{\otimes n} \xrightarrow{f} F(1)^{\otimes n} \to \Sigma\Omega F(1)^{\otimes n} \to 0$$

The map f can be identified with the usual Frobenius map which sends each element to its square. Its image consists of the span of those monomials $t_1^{2^{b_1}} \dots t_n^{2^{b_n}}$ such that each b_i is positive.

Consequently, $\Sigma\Omega F(1)^{\otimes n}$ can be identified with a submodule of

$$\oplus_{1 \leq i \leq n} F(1)^{\otimes i} \otimes \Sigma \mathbf{F}_2 \otimes F(1)^{\otimes n-i-1} \simeq \oplus_{1 \leq i \leq n} \Sigma F(1)^{\otimes n-1},$$

which is finitely generated by the inductive hypothesis.