18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Some Unstable Injectives (Lecture 16)

Let $\mathcal{U}$ denote the category of unstable modules over the Steenrod algebra $\mathcal{A}$. Then $\mathcal{U}$ has enough projective objects: that is, for every unstable $\mathcal{A}$-module M, there exists a surjection $P \to M$, where P is projective. For example, we can take $P = \oplus_{x \in M^n} F(n)$, equipped with its evident map to M.

The category $\mathcal{U}$ also has enough injective objects: that is, for every unstable $\mathcal{A}$-module M, there exists an injection $M \to I$, where I is injective. This is a general property of Grothendieck abelian categories (as demonstrated by Grothendieck). However, in the case of the category $\mathcal{U}$ we can verify this directly, by producing a large class of injective objects:

Proposition 1. *Let $n \geq 0$ be a nonnegative integer. Then there exists an unstable $\mathcal{A}$-module $J(n)$ equipped with a map $\chi : J(n)^n \to \mathbf{F}_2$ with the following universal property: for every unstable $\mathcal{A}$-module M, composition with χ induces a bijection*

$$\mathrm{Hom}_{\mathcal{A}}(M, J(n)) \to \mathrm{Hom}_{\mathbf{F}_2}(M^n, \mathbf{F}_2).$$

Proof. We sketch two different arguments.

First, the existence of $J(n)$ follows by abstract nonsense. If $\mathcal{C}$ is any category, then we say a functor $F : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$ is *representable* if there exists an object $X \in \mathcal{C}$ and a collection of bijections

$$F(C) \simeq \mathrm{Hom}_{\mathcal{C}}(C, X),$$

depending functorially on C. Any representable functor carries colimits in $\mathcal{C}$ to limits in $\mathcal{S}\mathrm{et}$ (essentially by definition). If $\mathcal{C}$ is a Grothendieck abelian category, then the converse holds (more generally, the converse holds whenever $\mathcal{C}$ is a *presentable* category). We apply this observation to the case $\mathcal{C} = \mathcal{U}$, and $F : \mathcal{U}^{op} \to \mathcal{S}\mathrm{et}$ is defined by the formula

$$M \mapsto (M^n)^\vee.$$

It is easy to see that F carries colimits to limits, so that F is representable by an unstable $\mathcal{A}$-module $J(n)$.

An alternative approach is to describe $J(n)$ directly. The universal property of $J(n)$ dictates its structure: for each integer k, we have

$$J(n)^k \simeq \mathrm{Hom}_{\mathcal{A}}(F(k), J(n)) \simeq (F(k)^n)^\vee.$$

For each $i \geq 0$, the map $\mathrm{Sq}^i : J(n)^k \to J(n)^{k+i}$ is dual to the map $F(k+i)^n \to F(k)^n$ induced by the map of unstable $\mathcal{A}$-modules $F(k+i) \to F(k)$ classified by the element $\mathrm{Sq}^i \nu_k \in F(k)^{k+i}$. It is not difficult to check that this endows

$$J(n) = \oplus_k J(n)^k = \oplus_k (F(k)^n)^\vee$$

with the structure of an unstable $\mathcal{A}$-module, and that this module has the desired universal property (exercise). $\square$

The $\mathcal{A}$-modules $J(n)$ are called *Brown-Gitler modules*, because they arise as the $\mathbf{F}_2$-homology of certain spectra called *Brown-Gitler spectra*. We will not use this description in this course.

For each $n \geq 0$, the Brown-Gitler module $J(n)$ represents the functor $M \mapsto (M^n)^\vee$. Since this functor is exact, the object $J(n) \in \mathcal{U}$ is injective.

Corollary 2. *The category $\mathcal{U}$ has enough injective objects.*

Proof. Let M be an unstable $\mathcal{A}$-module. To every map $f : M^n \to \mathbf{F}_2$, we can associate a map of $\mathcal{A}$-modules $M \to J(n)$. Taking the product over all pairs (n, f), we obtain a map

$$M \to \prod_{f : M^n \to \mathbf{F}_2} J(n).$$

This map is clearly injective. The right hand side is a product of Brown-Gitler modules, and therefore injective. □

Our next goal is to describe some other examples of injective objects in $\mathcal{U}$.

We have already met some other examples of injective objects of $\mathcal{U}$: namely, the cohomology rings $\mathrm{H}^*(BV)$, where V is a finite dimensional vector space over $\mathbf{F}_2$. These are very different from the Brown-Gitler modules $J(n)$. For example, for $n > 0$, the Brown-Gitler module $J(n)$ is *nilpotent*: that is, for every homogeneous element $x \in J(n)$, the sequence

$$x, \mathrm{Sq}^{\deg(x)} x, \mathrm{Sq}^{2\deg(x)} \mathrm{Sq}^{\deg(x)} x, \ldots$$

is eventually zero (since $J(n)$ vanishes in degrees $> n$). On the other hand, the cohomology ring $\mathrm{H}^*(BV)$ is isomorphic to a polynomial ring, and is therefore *reduced*: the map $x \mapsto \mathrm{Sq}^{\deg(x)} x$ is injective.

The injective objects $\mathrm{H}^*(BV$ have an unusual property: namely, the tensor product of any pair $\mathrm{H}^*(BV) \otimes \mathrm{H}^*(BW)$ isomorphic to $\mathrm{H}^*(B(V \oplus W))$, and is therefore again injective. In fact, the operation $M \mapsto \mathrm{H}^*(BV) \otimes M$ preserves injective objects in general. We wish to prove this in the case where M is a Brown-Gitler module. For this, we need to introduce some auxiliary constructions.

Proposition 3. *The inverse limit K of any sequence*

$$\ldots \to J(n_2) \to J(n_1) \to J(n_0)$$

of Brown-Gitler modules is injective as an unstable $\mathcal{A}$-module.

Proof. By definition, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(M, K) &\simeq \operatorname{proj\,lim} \mathrm{Hom}_{\mathcal{A}}(M, J(n_i)) \\ &\simeq \operatorname{proj\,lim} (M^{n_i})^\vee \\ &\simeq (\operatorname{inj\,lim} M^{n_i})^\vee. \end{aligned}$$

This is an exact functor, since it is dual to the exact functor

$$M \mapsto \operatorname{inj\,lim}(M^{n_0} \to M^{n_1} \to \ldots).$$

□

To apply Proposition 3, we need to understand maps between the Brown-Gitler modules $J(k)$. This is easy: by definition, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(J(m), J(n)) &\simeq (J(m)^n)^\vee \\ &\simeq \mathrm{Hom}_{\mathcal{A}}(F(n), J(m))^\vee \\ &\simeq ((F(n)^m)^\vee)^\vee \\ &\simeq F(n)^m \\ &\simeq \mathrm{Hom}_{\mathcal{A}}(F(m), F(n)) \end{aligned}$$

In particular, $\mathrm{Hom}_{\mathcal{A}}(J(m), J(n))$ has a basis consisting of Steenrod operations $\{\mathrm{Sq}^I\}$, where I is positive, admissible, $\deg(I) = m - n$, and the excess of I is $\leq n$. We will abuse notation and identify the elements $\mathrm{Sq}^I \in \mathcal{A}$ with the correponding maps between Brown-Gitler modules.

Definition 4. Let n be a nonnegative integer. The *Carlsson module* $K(n)$ is defined to be the inverse limit of the sequence

$$\ldots \to J(4n) \stackrel{\mathrm{Sq}^{2n}}{\to} J(2n) \stackrel{\mathrm{Sq}^{n}}{\to} J(n).$$

From Proposition 3 we immediately deduce:

Corollary 5. *For each $n \geq 0$, the Carlsson module $K(n)$ is an injective object of $\mathcal{U}$.*

From this description, we immediately deduce:

Proposition 6. *Let M be an unstable $\mathcal{A}$-module, and let n be a nonnegative integer. Then the canonical map $\Phi M \to M$ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(M, K(n)) \to \mathrm{Hom}_{\mathcal{A}}(\Phi M, K(n)).$$

Corollary 7. *Let M be an unstable $\mathcal{A}$-module, and let n be a nonnegative integer. Then $\mathrm{Hom}_{\mathcal{A}}(\Sigma M, K(n)) = 0$.*

Proof. This follows from Proposition 6, since the map $\Phi \Sigma M \to \Sigma M$ vanishes (this follows from the instability condition on M). □

An unstable $\mathcal{A}$-module M is *reduced* if the canonical map $f : \Phi M \to M$. In other words, M is reduced if $\mathrm{Sq}^{\deg x} x = 0$ implies that $x = 0$, for every homogeneous element $x \in M$. If M is an unstable $\mathcal{A}$-algebra, then the map $x \mapsto \mathrm{Sq}^{\deg x} x$ coincides with the map $x \mapsto x^2$, so that M is reduced if and only if it contains no nilpotent elements (this is the usual meaning of the term *reduced* in commutative algebra).

Corollary 8. *For every nonnegative integer n, the Carlsson module $K(n)$ is reduced.*

Proof. Let M denote the submodule of $K(n)$ generated by those homogeneous elements $x \in K(n)^k$ such that $\mathrm{Sq}^k x = 0$. Then the map $\Phi M \to M$ vanishes, so $M \simeq \Sigma \Omega M$. Applying Corollary 7, we conclude that the inclusion $M \subseteq K(n)$ is the zero map, so that $M = 0$. □

Suppose that M is a reduced unstable $\mathcal{A}$-module. Then any map $M \to J(n)$ factors through $K(n)$. Equivalently, any functional on M^n can be extended to the direct limit

$$M^n \stackrel{\mathrm{Sq}^{n}}{\to} M^{2n} \stackrel{\mathrm{Sq}^{2n}}{\to} M^{4n} \to \ldots;$$

this follows from the observation that M^n injects into this direct limit. Consequently, the embedding

$$M \to \prod_{f : M^n \to \mathbf{F}_2} J(n)$$

of Corollary 2 can be lifted to a map

$$M \to \prod_{f : M^n \to \mathbf{F}_2} K(n).$$

It is easy to see that this map is again injective. We have therefore proven:

Proposition 9. *Let M be a reduced unstable $\mathcal{A}$-module. Then there exists a monomorphism*

$$M \to \prod_{\alpha} K(n_{\alpha})$$

for some collection of nonnegative integers $\{n_{\alpha}\}$.

Corollary 10. *Let V be a finite dimensional vector space over $\mathbf{F}_2$. Then the unstable $\mathcal{A}$-module $\mathrm{H}^*(BV)$ is isomorphic to a direct summand of some product $\prod_{\alpha} K(n_{\alpha})$.*

Proof. The cohomology ring $\mathrm{H}^*(BV)$ is isomorphic to a polynomial ring $\mathbf{F}_2[t_1, \ldots, t_n]$, and therefore contains no nilpotent elements. Consequently, $\mathrm{H}^*(BV)$ is reduced as an unstable $\mathcal{U}$-module. Applying Proposition 9, we deduce the existence of a monomorphism

$$j : \mathrm{H}^*(BV) \to \prod_\alpha K(n_\alpha).$$

We saw earlier that the unstable $\mathcal{A}$-module $\mathrm{H}^*(BV)$ is injective. Consequently, the identity map $\mathrm{id} : \mathrm{H}^*(BV) \to \mathrm{H}^*(BV)$ can be extended to a map $p : \prod_\alpha K(n_\alpha) \to \mathrm{H}^*(BV)$, which is a left inverse to j. We therefore obtain a direct sum decomposition

$$\prod_\alpha K(n_\alpha) \simeq \mathrm{H}^*(BV) \oplus \ker(p).$$

□

Since the Brown-Gitler modules $J(k)$ are finite-dimensional in each degree, the operation $M \mapsto M \otimes J(k)$ preserves products. Consequently, we deduce the following:

Corollary 11. *Let V be a finite dimensional vector space over $\mathbf{F}_2$, and k a nonnegative integer. Then the tensor product*

$$\mathrm{H}^*(BV) \otimes J(k)$$

is a direct summand of some product

$$\prod_\alpha K(n_\alpha) \otimes J(k).$$

Consequently, to prove that a tensor product $\mathrm{H}^*(BV) \otimes J(k)$ is injective, it will suffice to show that each tensor product $K(n) \otimes J(k)$ is injective. We will return to this point next time.