Injectivity of Tensor Products (Lecture 17)

Our goal in this lecture is to prove the following result:

Theorem 1. *Let n and k be nonnegative integers. Then the tensor product $K(n) \otimes J(k)$ is an injective object in the category of unstable $\mathcal{A}$-modules.*

We begin with some general remarks. For every nonnegative integer p, the Brown-Gitler module $J(p)$ comes equipped with a canonical functional $J(p)^p \to \mathbf{F}_2$. Given a pair of integers $p, q \geq 0$, we obtain an induced map

$$(J(p) \otimes J(q))^{p+q} \to J(p)^p \otimes J(q)^q \to \mathbf{F}_2 \otimes F_2 \simeq \mathbf{F}_2,$$

which induces a map

$$\mu_{p,q} : J(p) \otimes J(q) \to J(p+q).$$

The proof of Theorem 1 depends on the following observation:

Lemma 2. *Fix nonnegative integers n, k, and a. Then the map*

$$\mu^a_{2^p n,k} : (J(2^p n) \otimes J(k))^a \to J(2^p n + k)^a$$

is an isomorphism for $p \gg 0$.

We now give the proof of Theorem 1, assuming Lemma 2. For each $m \geq 0$, let $f : J(2m) \to J(m)$ be the map of Brown-Gitler modules corresponding to the Steenrod operation Sq^m. For $0 \leq p \leq q$, let $F_{p,q} : J(2^q n) \to J(2^p n)$ denote the composition

$$J(2^q n) \xrightarrow{f} \ldots \xrightarrow{f} J(2^p n).$$

We will construct a sequence of integers $0 = p_0 < p_1 < p_2 < \ldots$ and maps $G_i : J(2^{p_{i+1}} n + k) \to J(2^{p_i} n + k)$ such that the diagrams

$$\begin{array}{ccc} J(2^{p_{i+1}} n) \otimes J(k) & \longrightarrow & J(2^{p_{i+1}} n + k) \\ \downarrow {\scriptstyle F_{p_{i+1},p_i} \otimes \mathrm{id}} & & \downarrow {\scriptstyle G_i} \\ J(2^{p_i} n) \otimes J(k) & \longrightarrow & J(2^{p_i} n + k) \end{array}$$

are commutative. In fact, the existence and uniqueness of G_i are clear as soon as the upper horizontal map is an isomorphism in degree $2^{p_i} n + k$. Lemma 2 implies that this is true provided that p_{i+1} is chosen large enough.

By definition, the Carlsson module $K(n)$ is defined to be the inverse limit of the sequence

$$\ldots \to J(4n) \to J(2n) \xrightarrow{f} J(n).$$

It can equally well be defined as the inverse limit of the subsequence

$$\ldots \to J(2^{p_2} n) \to J(2^{p_1} n) \to J(2^{p_0} n).$$

Since $J(k)$ is finite dimensional, we can identify $K(n) \otimes J(k)$ with the inverse limit of the sequence

$$\ldots \to J(2^{p_2}n) \otimes J(k) \to J(2^{p_1}n) \otimes J(k) \to J(2^{p_0}n) \otimes J(k).$$

The multiplication maps $\mu_{2^{p_i}n,k}$ determine a homomorphism from this inverse system to the inverse system

$$\ldots \to J(2^{p_2}n + k) \stackrel{G_1}{\to} J(2^{p_1}n + k) \stackrel{G_0}{\to} J(2^{p_0}n + k).$$

For every $a \geq 0$, Lemma 2 guarantees that $\mu_{2^{p_i}n,k}$ is an isomorphism in degree a for $i \gg 0$. Consequently, we get an isomorphism of inverse limits

$$K(n) \otimes J(k) \simeq \lim \{J(2^{p_i} + k)\}_{i \geq 0}.$$

In the last lecture, we saw that any inverse limit of Brown-Gitler modules is injective. It follows that $K(n) \otimes J(k)$ is injective, as desired.

We now turn to the proof of Lemma 2. The domain of $\mu^a_{2^p n,k}$ can be identified with the direct sum

$$\oplus_{a=a'+a''} J(2^p n)^{a'} \otimes J(k)^{a''}.$$

Recall that, for every pair of integers x and y, we have canonical isomorphisms

$$J(x)^y \simeq \mathrm{Hom}_{\mathcal{A}}(F(y), J(x)) = (F(y)^x)^\vee.$$

Using these isomorphisms, we can identify $\mu^a_{2^p n,k}$ with the dual of the canonical map

$$\phi : F(a)^{2^p n+k} \to \oplus_{a=a'+a''} F(a')^{2^p n} \otimes F(a'')^k.$$

Let us identify $F(m)$ with the subspace of the polynomial ring $\mathbf{F}_2[t_1, \ldots, t_m]$ consisting of symmetric additive polynomials. For each monomial $f = t_1^{i_1} \ldots t_m^{i_m}$, let $\sigma(f)$ denote the symmetrization of f as in Lecture 7, so that f appears in $\sigma(f)$ with multiplicity one. Then $F(a)^{2^p n+k}$ has a basis consisting of the symmetrizations of monomials of the form

$$t_1^{2^{i_1}} \ldots t_a^{2^{i_a}}$$

where $i_1 \leq i_2 \leq \ldots \leq i_a$, and $\sum 2^{i_j} = 2^p n + k$. If $p \gg 0$, then Lemma 3 below implies that there exists a unique $a'' \leq a$ such that

$$2^{i_1} + \ldots + 2^{i_{a''}} = k$$

$$2^{i_{a''+1}} + \ldots + 2^{i_a} = 2^p n.$$

We now observe that ϕ carries $\sigma(t_1^{2^{i_1}} \ldots t_a^{2^{i_a}})$ to the tensor product

$$\sigma(t_1^{2^{i_{a''+1}}} \ldots t_{a'}^{2^{i_a}}) \otimes \sigma(t_1^{2^{i_1}} \ldots t_{a''}^{2^{i_{a''}}}),$$

and that these tensor products form a basis for

$$\oplus_{a=a'+a''} F(a')^{2^p n} \otimes F(a'')^k.$$

It remains only to verify:

Lemma 3. *Fix nonnegative integers n, k, and a. Then for every sufficiently large integer p and every equation*

$$2^p n + k = 2^{i_1} + \ldots + 2^{i_a},$$

there exists a unique partition $\{1, \ldots, a\} = J \coprod J'$, such that

$$2^p n = \sum_{j \in J} 2^{i_j}$$

$$k = \sum_{j \in J'} 2^{i_j}.$$

Proof. Let 2^b be the smallest power of 2 larger than k. We will prove that the assertion is true provided that Let $J_0 = \{1 \leq j \leq a : i_j > b\}$, and let $J_0' = \{1 \leq j \leq a : i_j \leq b\}$.

It is clear that any decomposition $\{1, \ldots, a\} = J \coprod J'$ must satisfy $J' \subseteq J_0'$: otherwise, we have

$$\sum_{j \in J'} 2^{i_j} > 2^b \geq k.$$

We will show that $\sum_{j \in J_0'} 2^{i_j} = k$ provided that p is sufficiently large. Then the containment $J' \subseteq J_0'$ forces $J' = J_0'$, so that (J_0, J_0') is the unique partition with the desired property.

Since every base 2-digit of k must appear in the sum $2^{i_1} + \ldots + 2^{i_a}$, we deduce that $\sum_{j \in J_0'} 2^{i_j} \geq k$. Let $k' = (\sum_{j \in J_0'} 2^{i_j}) - k$. We wish to prove that $k' = 0$. Suppose otherwise. We note that $k' \leq a2^b$. Moreover, the sum

$$k' + \sum_{j \in J_0} 2^{i_j} = 2^p n$$

is divisible by 2^p. It follows that the largest nonzero digit of k' is at least 2^{p-a}. On the other hand, k' is bounded above by $a2^b$, which is $< 2^{p-a}$ provided that $p \gg 0$. $\square$