18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Lannes' T-functor (Lecture 18)

In this lecture we will introduce Lannes' T-functor and verify some of its basic properties. We begin with a definition.

Definition 1. Let M be an unstable $\mathcal{A}$-module. We will say that M is *finite type* if each graded piece M^n is finite dimensional.

Proposition 2. *Let M be an unstable $\mathcal{A}$-module of finite type. Then the functor*

$$N \mapsto M \otimes N$$

admits a left adjoint, which we will (temporarily) denote by D_M.

Proof. According to the adjoint functor theorem, the main thing we need to check is that the functor $N \mapsto M \otimes N$ preserves limits. This follows immediately from the assumption that M is of finite type. Thus, the existence of D_M follows from abstract categorical nonsense.

We will sketch another proof which described how to compute D_M in practice. We first describe the value of D_M on a free unstable $\mathcal{A}$-module $F(n)$. By definition, we need

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}}(D_M F(n), N) &\simeq \mathrm{Hom}_{\mathcal{A}}(F(n), M \otimes N) \\
&\simeq (M \otimes N)^n \\
&\simeq \oplus_{n=n'+n''} M^{n'} \otimes N^{n''} \\
&\simeq \oplus_{n=n'+n''} \mathrm{Hom}_{\mathcal{A}}((M^{n'})^\vee \otimes F(n''), N).
\end{aligned}$$

This is obviously satisfied if we take $D_M F_n = \oplus_{n=n'+n''} (M^{n'})^\vee \otimes F(n'')$.

We now extend the definition of D_M to the category of all unstable $\mathcal{A}$-modules in such a way that D_M commutes with colimits. To define $D_M(N)$, we choose an exact sequence

$$\oplus_\beta F(n_\beta) \to \oplus_\alpha F(n_\alpha) \to N \to 0,$$

and define $D_M(N)$ to be the cokernel of the induced map

$$\oplus_\beta D_M F(n_\beta) \to \oplus_\alpha D_M F(n_\alpha).$$

It is easy to verify that this cokernel has the desired universal property. □

Example 3. Let $M = \Sigma \mathbf{F}_2$, so that the functor $N \mapsto N \otimes M$ is equivalent to the suspension functor Σ. Then D_M is isomorphic to the functor Ω studied in a previous lecture.

Definition 4. Let V be a finite dimensional vector space over $\mathbf{F}_2$. Then the cohomology ring $M = \mathrm{H}^*(BV)$ is an unstable $\mathcal{A}$-module of finite type. We will denote the functor D_M in this case by T_V. The functor T_V is called *Lannes' T-functor*.

Remark 5. Let X be an arbitrary topological space, and let X^{BV} denote the space of maps from BV into X. We have a canonical evaluation map

$$X^{BV} \times BV \to X.$$

This induces a pullback map on cohomology

$$\mathrm{H}^*(X) \to \mathrm{H}^*(X^{BV} \times BV) \simeq \mathrm{H}^*(X^{BV}) \otimes \mathrm{H}^*(BV).$$

This determines an adjoint map

$$T_V \, \mathrm{H}^*(X) \to \mathrm{H}^*(X^{BV}).$$

We will later see that this adjoint map is often an isomorphism. Therefore, Lannes' T-functor provides a purely algebraic mechanism for understanding the cohomology of mapping spaces.

For now, we will be content to establish some of the basic formal properties of the functor T_V. We begin with the following result, which is a reformulation of the work of the previous lectures:

Proposition 6. *Let V be a finite dimensional vector space over $\mathbf{F}_2$. Then the functor T_V is exact.*

Proof. Choose an exact sequence

$$0 \to M' \to M \to M'' \to 0.$$

We wish to show that the induced sequence

$$0 \to T_V M' \to T_V M \to T_V M'' \to 0$$

is also exact. It will suffice to show that this sequence is exact in each degree. For this, we need only show that for each $n \geq 0$, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(T_V M'', J(n)) \to \mathrm{Hom}_{\mathcal{A}}(T_V M, J(n)) \to \mathrm{Hom}_{\mathcal{A}}(T_V M', J(n)) \to 0.$$

Invoking the definition of T_V, we can rewrite this sequence as

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M'', J(n) \otimes \mathrm{H}^*(BV)) \to \mathrm{Hom}_{\mathcal{A}}(M, J(n) \otimes \mathrm{H}^*(BV)) \to \mathrm{Hom}_{\mathcal{A}}(M', J(n) \otimes \mathrm{H}^*(BV)) \to 0.$$

The exactness now follows from the injectivity of the object $J(n) \otimes \mathrm{H}^*(BV)$. □

We now discuss the relationship between the functor T_V and suspension. Recall that the suspension functor Σ can be identified with the functor $M \mapsto M \otimes \Sigma(\mathbf{F}_2)$. Consequently, the functors Σ and $M \mapsto M \otimes \mathrm{H}^*(V)$ commute with one another: composing them in either order yields the functor

$$M \mapsto M \otimes \Sigma \, \mathrm{H}^*(V).$$

Passing to left adjoints, we get a canonical isomorphism of functors

$$T_V \Omega \simeq \Omega T_V.$$

This isomorphism induces a natural transformation

$$\begin{aligned} T_V \Sigma &\to \Sigma \Omega T_V \Sigma \\ &\simeq \Sigma T_V \Omega \Sigma \\ &\to \Sigma T_V \end{aligned}$$

We wish to prove that this map is also an isomorphism of functors. For this, we first construct a *right* adjoint to the functor Σ. The functor Σ commutes with all limits and colimits, and therefore admits a right adjoint by the adjoint functor theorem. However, we can describe this right adjoint more concretely.

Proposition 7. *Let M be an unstable module over the Steenrod algebra, and let M' denote the subspace of M spanned by those homogeneous elements x such that $\mathrm{Sq}^{\deg x} x = 0$. Then:*

(1) *M' is a $\mathcal{A}$-submodule of M.*

(2) *M' has the form $\Sigma\widetilde{\Sigma}M$, for some $\mathcal{A}$-module $\widetilde{\Sigma}M$.*

(3) *For every unstable $\mathcal{A}$-module N, the inclusion $\Sigma\widetilde{\Sigma}M \subseteq M$ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(N, \widetilde{\Sigma}M) \to \mathrm{Hom}_{\mathcal{A}}(\Sigma N, M).$$

(4) *The functor $M \mapsto \widetilde{\Sigma}M$ is right adjoint to the suspension functor.*

Proof. To prove (1), we observe that M' can be identified (as a vector space) with the kernel of the canonical map $f : \Phi M \to M$. Since f is a map of $\mathcal{A}$-modules, we deduce that M' is stable under the action of the Steenrod algebra on ΦM. In other words, M' is a $\mathcal{A}$-submodule of M, where $\mathcal{A}$ acts on M via the composition

$$\mathcal{A} \xrightarrow{V} \mathcal{A} \to \mathrm{End}(M)$$

where V is the Verschiebung map $\mathrm{Sq}^n \mapsto \mathrm{Sq}^{\frac{n}{2}}$. Since V is surjective, we conclude that M' is also stable under the usual action of $\mathcal{A}$ on M.

To prove (2), we observe that the map $\Phi M' \to M'$, vanishes, so that we obtain an isomorphism $M' \to \Sigma\Omega M'$. We can now take $\widetilde{\Sigma}M = \Omega M'$.

To prove (3), we observe that Σ is fully faithful, so we have an isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(N, \widetilde{\Sigma}M) \simeq \mathrm{Hom}_{\mathcal{A}}(\Sigma N, \Sigma\widetilde{\Sigma}M) = \mathrm{Hom}_{\mathcal{A}}(\Sigma N, M').$$

To complete the proof, it will suffice to show that $\mathrm{Hom}_{\mathcal{A}}(\Sigma N, M') = \mathrm{Hom}_{\mathcal{A}}(\Sigma N, M)$: in other words, that every map from ΣN into M factors through M'. This follows from the observation that for $x \in \Sigma N$, we have $\mathrm{Sq}^{\deg x}(x) = 0$.

Assertion (4) is an immediate consequence of (3). □

Proposition 8. *Let V be a finite dimensional $\mathbf{F}_2$-vector space. The natural transformation*

$$T_V \Sigma \to \Sigma T_V$$

is an isomorphism of functors.

Proof. It will suffice to prove that the induced map between right adjoints is an isomorphism of functors. In other words, we must show that for every unstable $\mathcal{A}$-module M, the induced map

$$\mathrm{H}^*(BV) \otimes \widetilde{\Sigma}M \to \widetilde{\Sigma}(\mathrm{H}^*(BV) \otimes M)$$

is an isomorphism. Unwinding the definitions, we must show that the map

$$i : \mathrm{H}^*(BV) \otimes M' \to (\mathrm{H}^*(BV) \otimes M)'$$

is an isomorphism, where M' denotes the submodule of M defined in Proposition 7 and $(\mathrm{H}^*(BV) \otimes M)'$ is defined similarly. The injectivity of i is obvious, since both sides can be identified with submodules of $\mathrm{H}^*(BV) \otimes M$.

To prove the surjectivity, let us define by f_M the Frobenius map $x \mapsto \mathrm{Sq}^{\deg x}(x)$. It is easy to see that for every pair of unstable $\mathcal{A}$-modules M and N, we have

$$f_{M \otimes N} = f_M \otimes f_N,$$

so that

$$\ker(f_{M\otimes N}) = (\ker(f_M) \otimes N) + (M \otimes \ker(f_N)).$$

In particular, if N is reduced, then $\ker f_N = 0$, so

$$\ker(f_{M\otimes n}) = \ker(f_M) \otimes N.$$

We now conclude the proof by observing that $\mathrm{H}^*(BV) \simeq \mathbf{F}_2[t_1, \ldots, t_k]$ is reduced. $\square$